

Combinatorial and hybrid principles for σ -directed families of countable sets modulo finite

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Abstract

We consider strong combinatorial principles for σ -directed families of countable sets in the ordering by inclusion modulo finite, e.g. P -ideals of countable sets. We try for principles as strong as possible while remaining compatible with CH, and we also consider principles compatible with the existence of nonspecial Aronszajn trees. The main thrust is towards abstract principles with game theoretic formulations. Some of these principles are purely combinatorial, while the ultimate principles are primarily combinatorial but also have aspects of forcing axioms.

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1. INTRODUCTION

Very often combinatorial phenomena occurring at the first uncountable ordinal can be expressed by sentences in the second level of the Lévy hierarchy over the structure (H_{\aleph_2}, \in) or more generally the structure $(H_{\aleph_2}, \in, \text{NS})$, where NS denotes the ideal of nonstationary subsets of ω_1 . For example, the continuum hypothesis (CH) is Σ_2 (i.e. of the form $\ulcorner \exists x \forall y \varphi(x, y) \urcorner$ where the formula φ has no unbounded quantifiers)¹ over (H_{\aleph_2}, \in) and Souslin's hypothesis (SH) is Π_2 (i.e. of the form $\ulcorner \forall x \exists y \varphi(x, y) \urcorner$) over (H_{\aleph_2}, \in) . And there is a spectrum of axioms with axioms such as $V = L$ at one end, which tend to minimize the collection of Π_2 sentences that hold over $(H_{\aleph_2}, \in, \text{NS})$ (and thus maximize the Σ_2 sentences), and axioms such as Martin's Maximum (MM) at the other end, maximizing the Π_2 sentences in the theory of $(H_{\aleph_2}, \in, \text{NS})$. Similarly, there are Π_2 -poor (i.e. Σ_2 -rich) models like L at one end of the spectrum and Π_2 -rich models such as the optimal $L(\mathbb{R})^{\mathbb{P}_{\max}}$ at the other end.

In the present article we are interested in the latter end of the spectrum where Π_2 sentences are maximized in the theory of $(H_{\aleph_2}, \in, \text{NS})$. When one wants to demonstrate $H_{\aleph_2} \models \phi$ for some Π_2 sentence ϕ , for example in the typical case of establishing a consistency result, there are a number of options. One can try and show that some forcing axiom such as Martin's Axiom (MA), or some more powerful forcing axiom such as the proper forcing axiom (PFA) or MM, implies $H_{\aleph_2} \models \phi$ (i.e. internal forcing). Or one can try to directly force the truth of ϕ over H_{\aleph_2} with a forcing notion tailored to the sentence ϕ (i.e. external forcing). Alternatively, the theory of \mathbb{P}_{\max} can be applied for this purpose.

There is yet another approach, which is to find combinatorial statements so strong that they entail numerous Π_2 statements over $(H_{\aleph_2}, \in, \text{NS})$ including the one of interest. There are many examples in the literature demonstrating the power of this method. What we are getting at is that if we have some fixed strong combinatorial principle (P), then a proof that $(P) \rightarrow (H_{\aleph_2}, \in, \text{NS}) \models \phi$ is generally going to be a purely combinatorial argument that is relatively short in length, whereas applying either internal or external forcing requires the construction of a poset along with the necessary density arguments which is often lengthier.

The advantage is even far more pronounced when we want to produce a model where $H_{\aleph_2} \models \phi$, but at the same time making sure some other statement ψ holds. For example, we shall need to obtain models of various Π_2 sentences over $(H_{\aleph_2}, \in, \text{NS})$ subject to the condition that CH holds. In this situation, internal forcing may not be an option. In fact, to date there are only a few known forcing axioms compatible with CH ([FMS88, §3], [She98, VII, §2], [She98, XVII, §2], [She00a]) one of which we shall use in section 4. In many cases however, for example in section 4 when establishing one of the strong combinatorial principles in conjunction with $\psi = \ulcorner \text{CH} \wedge \text{there exists a nonspecial Aronszajn tree} \urcorner$, there is no known suitable forcing axiom, and external forcing is the only

¹ More precisely, it can be expressed as a Σ_2 statement, e.g. $\ulcorner \text{there exists a function on } \omega_1 \text{ whose range includes all of the real numbers } \mathbb{R} \urcorner$.

viable option—indeed, to the author’s knowledge no analogue of \mathbb{P}_{\max} has ever been successfully applied to produce models of CH. What is more, the method of external forcing to produce such a model of CH usually involves iterated forcing constructions that tend to be difficult and often long and tedious.

To make an analogy with computer programming, external (typically iterated) forcing and \mathbb{P}_{\max} are like machine languages giving maximum control over the resulting model, comparable to a low-level programming language that gives maximum control over the computer. On the other hand, internal forcing, and even more so applying some strong combinatorial principle, are like high-level programming languages where there may be several layers of abstraction (perhaps one or two for internal forcing and two or three for combinatorial principles) and the model can be constructed using a more human language and with considerably less effort. In the latter case, the underlying low-level arguments like iterated forcing or various \mathbb{P}_{\max} constructions can be safely ignored because they have already been done to obtain the forcing axiom or the combinatorial principle. This is analogous to the fact that high-level computer languages are normally automatically compiled into low-level languages that the computer can understand.

The first combinatorial principle that we are aware of and that entails a significant amount of the consistent Π_2 theory over H_{\aleph_2} is a Ramsey theoretic statement due to Todorćević written as $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ in the 1983 paper [Tod83]. There are surely many others with earlier dates, but we must draw the line somewhere as to what constitutes a ‘strong’ principle. We believe that the following two much stronger principles (A) and (A*) below are due to Todorćević, although the first place in the literature where we were able to find them is in the joint paper [AT97] appearing in 1997. (It is claimed in a recent article [Tod06, Remark 2] of Todorćević that the dichotomy (A) is just a restatement of his Ramsey theoretic principle appearing in the 1985 paper [Tod85], denoted there as (*), which is itself a slightly strengthened version of $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. However, this is either an error or a far-fetched exaggeration.) We need some notation and terminology before introducing these principles, both of which are dichotomies.

Notation 1.1. We use the standard set theoretic notation $[A]^\lambda$, $[A]^{\leq \lambda}$ and $[A]^{< \lambda}$ to denote the collection of subsets $B \subseteq A$ of cardinality $|B| = \lambda$, the collection of subsets $B \subseteq A$ of cardinality less than or equal to λ (i.e. $|B| \leq \lambda$), and the collection of subsets of cardinality less than λ , respectively. We write $\text{Fin}(A)$ for the set of all finite subsets of A .

We use the standard order theoretic notation $\downarrow \mathcal{H}$ to denote the downwards closure of \mathcal{H} in the inclusion order, i.e. $\downarrow \mathcal{H} = \bigcup_{x \in \mathcal{H}} \mathcal{P}(x)$. Assume $\mathcal{H} \subseteq \mathcal{I}$ for some set \mathcal{I} . When we want to take the downwards closure with respect to some other quasi ordering \lesssim of \mathcal{I} , we denote it by $\downarrow(\mathcal{H}, \lesssim)$.

Terminology 1. Let \mathcal{H} be a family of subsets of some fixed set S . We say that a subset $A \subseteq S$ is *locally in \mathcal{H}* if all of its countable subsets are members of \mathcal{H} ,

symbolically $[A]^{\leq \aleph_0} \subseteq \mathcal{H}$. We say that $B \subseteq S$ is *orthogonal* to \mathcal{H} , expressed symbolically as $B \perp \mathcal{H}$, if $B \cap x$ is finite for all $x \in \mathcal{H}$.

By noting that for A infinite, A is locally in $\downarrow \mathcal{H}$ is equivalent to

$$[A]^{\aleph_0} \subseteq \downarrow \mathcal{H}, \quad (1)$$

while B is orthogonal to \mathcal{H} is equivalent to

$$[B]^{\aleph_0} \cap \downarrow \mathcal{H} = \emptyset, \quad (2)$$

we can regard “locally in” and “orthogonal to” as dual notions.

There is however a nonequivalent dualization of orthogonality. We will consider the *almost inclusion* quasi order \subseteq^* on any power set, where $x \subseteq^* y$ if $x \setminus y$ is finite. An equivalent formulation of equation (2) is $b \subseteq^* y^c$ for all $y \in \mathcal{H}$, for all $b \in [B]^{\aleph_0}$. This can be dualized as $a \subseteq^* y$ for some $y \in \mathcal{H}$, for all $a \in [A]^{\aleph_0}$, or equivalently $[A]^{\leq \aleph_0} \subseteq \downarrow(\mathcal{H}, \subseteq^*)$. The latter is not equivalent to equation (1). We come back to this point just below.

Terminology 2. Recall that a subset of A of a quasi order (Q, \leq) is *directed* if every pair of elements $a, b \in A$ has an upper bound $c \in A$, i.e. $a, b \leq c$. More generally, for a cardinal λ , A is λ -*directed* if every subset of A of cardinality less than λ has an upper bound in A . Thus directed and \aleph_0 -directed are the same notion. σ -*directed* is the same thing as \aleph_1 -directed.

- (A) Let S be a set. For every directed subfamily \mathcal{H} of $([S]^{\leq \aleph_0}, \subseteq^*)$ of cardinality at most \aleph_1 , either
- (1) S has a countable decomposition into singletons and pieces locally in $\downarrow \mathcal{H}$, or
 - (2) there exists an uncountable subset of S orthogonal to \mathcal{H} .

The following dichotomy is the dual of (A) in the sense that the roles of the uncountable subset of S and the countable decomposition have been swapped.

- (A*) Let S be a set. For every directed subfamily \mathcal{H} of $([S]^{\leq \aleph_0}, \subseteq^*)$ of cardinality at most \aleph_1 , either
- (1) there exists an uncountable subset of S locally in $\downarrow \mathcal{H}$, or
 - (2) S has a countable decomposition into countably many pieces orthogonal to \mathcal{H} .

Remark 1.2. In the dichotomies appearing in [AT97], an \aleph_1 -generated ideal \mathcal{I} is specified rather than \mathcal{H} . If we were to use the weaker dual notion to orthogonality mentioned above, where $\downarrow \mathcal{H}$ is replaced with $\downarrow(\mathcal{H}, \subseteq^*)$, then $\downarrow(\mathcal{H}, \subseteq^*)$ is the ideal generated by $\mathcal{H} \cup \text{Fin}(S)$, and thus with this modification our versions of the dichotomies become the same as the originals, and in particular one does not need to mention the singletons in the first alternative (1) of the principle (A). In any case, while our statements (A) and (A*) are formally stronger than the originals, we will see in remark 2.7 that they are in fact equivalent.

The reason why we do not specify that \mathcal{H} is closed under subsets rather than using the notation $\downarrow \mathcal{H}$ will become apparent later.

The following result is discussed in [AT97].

Theorem (Todorćević). *PFA implies (A) and (A*).*

Both of the dichotomies (A) and (A*) are also known to negate the continuum hypothesis (cf. [AT97]).

The next major development in the domain of strong combinatorial principles took place in 1995 and was rather exciting. A combinatorial principle was formulated by Todorćević, namely the case $\theta = \omega_1$ of the principle (*) below, strong enough to imply many of the standard consequences of MA_{\aleph_1} and PFA such as Souslin's Hypothesis (SH), *all (ω_1, ω_1) gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ are indestructible* and more; and yet, as established by Abraham, consistent with CH.

- (*) Let θ be an ordinal.² For every σ -directed subfamily \mathcal{H} of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, either
- (1) there exists an uncountable $X \subseteq \theta$ locally in $\downarrow \mathcal{H}$, or
 - (2) θ can be decomposed into countably many pieces orthogonal to \mathcal{H} .

Remark 1.2 applies here as well. We have generalized the principle (*) so that it applies to arbitrary σ -directed families rather than just P -ideals. In this case the strengthening is purely formal (cf. lemma 2.6). The principle $(*)_{\omega_1}$, i.e. (*) with $\theta = \omega_1$, was then generalized to arbitrary θ by Todorćević in [Tod00]. Let us note here that for any ordinal of cardinality \aleph_2 or greater, this principle has large cardinal strength (see e.g. [Tod00], [Hir07b]). As we are mainly interested in the theory of $(H_{\aleph_2}, \in, \text{NS})$, we consider $\theta = \omega_1$ to be the most important case, and $(*)_{\omega_1}$ has no large cardinal strength as was demonstrated in [AT97].

1.1. Description of results. The independence of Souslin's Hypothesis, and Jensen's theorem on its consistency with CH, was a milestone in set theory; see for example the book [DJ74] devoted to this. Jensen moreover established that the stronger statement *all Aronszajn trees are special* is consistent with CH. Now the proof in [AT97] of the consistency of $(*)_{\omega_1}$ with CH and of the fact that $(*)_{\omega_1}$ implies SH, constituted the shortest proof to date of Jensen's theorem $\text{Con}(\text{CH} + \text{SH})$ —and we should also note that it relied heavily on the theory of Shelah [She82] developed in part for giving a more reasonable proof of Jensen's theorem. Hence this begged the following question.

Question (Abraham–Todorćević, 1995). *Does (*) imply that all Aronszajn trees are special?*

We give the expected negative answer here:

Theorem 1.1. *The conjunction of $(*)_{\omega_1}$ and CH is consistent, relative to ZFC, with the existence of a nonspecial Aronszajn tree. The conjunction of (*) and CH is consistent relative to a supercompact cardinal with the existence of a nonspecial Aronszajn tree.*

This can be interpreted as an indication that (*) is deficient for the study of trees. For the good behaviour of trees, the optimal principles for σ -directed

² Henceforth we shall work exclusively with families of sets of ordinals. Of course, the ordinal structure plays no role in this particular principle (*), and thus any set can be substituted for θ .

subfamilies of $[\theta]^{\leq \aleph_0}$ seem to be the dual $(*)^\partial$ of $(*)$ and the club variation (\star_c) of $(*)$ described below. The dualization of $(*)$ for arbitrary ordinals involves subtleties beyond the scope of this paper. Thus we just describe the dual of $(*)_{\omega_1}$.

- $(*)_{\omega_1}^\partial$ For every σ -directed family \mathcal{H} of countable subsets of ω_1 , either
- (1) there is a countable decomposition of ω_1 into singletons and sets locally in $\downarrow \mathcal{H}$, or
 - (2) there is an uncountable subset of ω_1 orthogonal to \mathcal{H} .

The proof that $(*) \rightarrow \text{SH}$ is easily adapted to show $(*)_{\omega_1}^\partial \rightarrow \text{SH}^+$, i.e. $(*)_{\omega_1}^\partial$ implies all Aronszajn trees are special (this follows immediately from lemmas 5.8 and 5.9 and corollary 5.10.1). The consistency of $(*)^\partial$ with CH was an open problem of Abraham–Todorćević, and the theory developed in the present paper is applied in [Hir07c] to obtain a positive solution.

The principle (\star_c) is a variation of $(*)$ due to Eisworth from 1997 (see [She00b, Question 2.17]), at least in the most important case $\theta = \omega_1$, and its consistency with CH remains an open problem.

- (\star_c) Let θ be an ordinal with uncountable cofinality. For every σ -directed subfamily \mathcal{H} of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, either
- (1) there is closed uncountable subset of θ locally in $\downarrow \mathcal{H}$, or
 - (2) there is a stationary subset of θ orthogonal to \mathcal{H} .

It is proved by the author in [Hir07b] that $(\star_c) \rightarrow \text{SH}^+$, and our interest there was that it also implies that SH^+ holds in any random forcing extension. (\star_c) is shown to be a consequence of PFA in [Hir07b], and this was first done by Eisworth for $\theta = \omega_1$.

The present paper evolved from the joint work [AH07] of the author with Abraham on the problem of whether all Aronszajn trees being *nearly special* implies that all Aronszajn trees are in fact special. Let us say that an ω_1 -tree T is nearly special if every stationary subset $S \subseteq \omega_1$ has a stationary subset $S' \subseteq S$ such that the subtree of T formed by the levels in S' is special. After obtaining a negative answer to this tree problem in [AH07], it was clear that the methods were relevant to the Abraham–Todorćević question (cf. page 5).

In fact, the strong answer obtained in [AH07] (cf. page 53) naturally led to a new dichotomy.

- (\star_s) Let θ be an ordinal of uncountable cofinality. Then there exists a maximal antichain \mathcal{A} of $(\text{NS}_\theta^+, \subseteq)$ such that for every σ -directed subfamily \mathcal{H} of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, either
- (1) every $S \in \mathcal{A}$ has an uncountable relatively closed $C \subseteq S$ locally in $\downarrow \mathcal{H}$,
or
 - (2) there exists a stationary subset of θ orthogonal to \mathcal{H} ,

where NS_θ^+ denotes the coideal of nonstationary subsets of θ . We will show that $(\star_s)_{\omega_1}$ implies that all Aronszajn trees are nearly special in section 5.2. On the other hand, we shall see that $(\star_s)_{\omega_1} \nrightarrow \text{SH}^+$, and thus the main result of [AH07] is also established here. Furthermore, in theorem 4.7 we obtain a

model of CH, that satisfies both $(*)_{\omega_1}$ and $(*)_s$ simultaneously, but also has a nonspecial Aronszajn tree in it.

Thus this paper is to a certain extent a generalization of the results in [AH07] from the realm of ω_1 -trees to the realm of σ -directed families of countable sets of ordinals quasi ordered by almost inclusion. This abstraction has another advantage not discussed in the introduction. A comparison shows that the proofs of the more general and thus stronger results for abstract families tend to be considerably shorter and less involved than the corresponding results for the concrete objects, in this case trees. On the other hand, it should be noted that the results here certainly do *not* supercede those in [AH07]. In many situations concrete objects are preferable to abstract ones and can give a better view of the mathematical argumentation. Moreover, in the paper [AH07] the methods there have an interesting application to bases of Aronszajn trees, that does not appear to readily fit into the present framework.

We recently noticed (after $(*)_s$ was formulated) that the principle $(*)_s$ is related to recent work in [EN07]. Consider the following weakening of $(*)_c$.

- $(*)_s$ Every σ -directed subfamily \mathcal{H} of $([\omega_1]^{\leq \aleph_0}, \subseteq^*)$ has either
- (1) a stationary subset of ω_1 locally in $\downarrow \mathcal{H}$, or
 - (2) a stationary subset of ω_1 orthogonal to \mathcal{H} .

Thus $(*)_s \rightarrow (*_c)$. This dichotomy is named P_{22} in [EN07] and shown there to be consistent with CH.

The developments just discussed can be also be observed from the viewpoint described in [SZ99]. Since the actual definition is well beyond the scope of this paper, we roughly describe a statement ϕ as Π_2 -compact (cf. [SZ99]) if there is no pair ψ_0 and ψ_1 of Π_2 sentences, possibly with the unary symbol NS, such that both $(H_{\aleph_2}, \in, \text{NS}) \models \ulcorner \psi_0 \urcorner \wedge \phi$ and $(H_{\aleph_2}, \in, \text{NS}) \models \ulcorner \psi_1 \urcorner \wedge \phi$ are consistent, yet $(H_{\aleph_2}, \in, \text{NS}) \models \ulcorner \psi_0 \wedge \psi_1 \urcorner \rightarrow \neg \phi$ (thus we are not relativizing ϕ to H_{\aleph_2}). Again speaking roughly, any Π_2 -compact sentence ϕ , according to the formulation in [SZ99], can be satisfied by some model $M \models \phi$ that also satisfies every Π_2 sentence ψ over $(H_{\aleph_2}, \in, \text{NS})$ that can be forced to be true over $(H_{\aleph_2}, \in, \text{NS})$, in conjunction with ϕ (in the presence of some large cardinals).

It is our understanding that the unanswered question of whether the continuum hypothesis is Π_2 -compact (e.g. [Woo99, Ch. 10,11]) is a substantial obstacle to continued progress on settling the continuum problem. This question is also very closely related to some of the questions of Shelah in [She00b, §2]. Finding stronger and stronger combinatorial principles compatible with CH is highly relevant to determining the Π_2 -compactness of CH. For example, if we found two combinatorial principles compatible with CH whose conjunction negates CH then we will have proved it non- Π_2 -compact.

To our knowledge the Π_2 -compactness of Souslin's hypothesis remains an open problem, although results in [SZ99] and [Lar99] suggest a negative answer. We have no idea whether the related question about the Π_2 -compactness of the existence of a nonspecial Aronszajn tree has ever been considered.

Question 1. *Is \lceil there exists a nonspecial Aronszajn tree \rceil a Π_2 -compact statement? How about $\neg\text{SH}^+ \wedge \text{CH}$?*

In striving for strong combinatorial principles such as $(*)_s$ compatible $\neg\text{SH}^+ \wedge \text{CH}$ we are working towards an answer to question 1.

We began by using games to facilitate the proof of the consistency of $(*)_s$, and this was highly successful because it allowed us to produce e.g. models of both $(*)$ and $(*)_s$ while only arguing once for properties such as α -properness, the properness isomorphism and so forth. However, they ended up being absorbed into the theory itself and led to new and stronger combinatorial principles based on these games. This resulted in the addition of a whole new layer of abstraction.

Until this work, proving each variation of the original principle $(*)$ to be consistent with CH required repeating long arguments with slight modifications according to the particular variation. Moreover, within a given variation, long arguments are again repeated, particularly between α -properness and the proper isomorphism condition. It is our hope that this additional abstraction will result in reduced redundancy in proving other combinatorial principles compatible with CH. This has already been realized in [Hir07c].

Here is one of the principles without definitions to give some idea of the concept.

- (*) Let $(\mathcal{F}, \mathcal{H})$ be a pair of subfamilies of $[\theta]^{\leq \aleph_0}$ for some ordinal θ , with \mathcal{F} closed under finite reductions. If \mathcal{H} is \mathcal{F} -extendable, and Complete has a forward nonlosing strategy in the parameterized game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, ψ -globally for some $\psi \rightarrow \psi_{\min}$, then there exists an uncountable $X \subseteq \theta$ such that every proper initial segment of X is in $\downarrow(\mathcal{H}, \sqsubseteq)$.

The games all take as parameters a pair $(\mathcal{F}, \mathcal{H})$ of families of countable sets of ordinals. These principles all assert the existence of an uncountable set all of whose proper initial segments are in $\downarrow(\mathcal{H}, \sqsubseteq)$. For example, we can obtain the principle $(*)$ by inputting $(\downarrow\mathcal{H}, \downarrow\mathcal{H})$ into the appropriate game theoretic combinatorial principle, and then verifying that a certain player in the corresponding class of games has winning strategy. This drastically reduces the amount of work needed to obtain a model of $(*)$ and CH ‘from scratch’. Of course the consistency of the game theoretic principle must be established first, and this is done in section 4.

There are two main parameterized games involved. One of the games $\mathfrak{D}_{\text{cmp}}$ is purely combinatorial and thus leads to purely combinatorial principles. There is one class \mathcal{R} of forcing notions associated with pairs $(\mathcal{F}, \mathcal{H})$ of families of countable sets of ordinals, introduced in section 3.1. In the second parameterized game $\mathfrak{D}_{\text{gen}}$, the outcome of the game depends on a genericity condition in the poset $\mathcal{R}(\mathcal{F}, \mathcal{H})$. Thus the corresponding principles, e.g. $(*)$ above, have aspects of both a combinatorial principle and a forcing axiom, and thus are hybrid principles. These seem to us to still have much more the essence of a combinatorial principle as opposed to a forcing axiom.

A number of questions concerning these principles are posed in section 4. We also make a comparison with the principle (A) in section 5.1. Specifically, we

found the “ \aleph_1 ” in the two dichotomies (A) and (A*) for arbitrary S a bit out of place in a combinatorial principle, and more like something we would expect in a forcing axiom (of course in the most important case $S = \omega_1$ it is quite natural, and disappears when we express the statement over H_{\aleph_2}). We suggest a kind of remedy for this in section 5.1, and strengthen Todorćević’s theorem on page 5 in doing so.

1.2. Credits and acknowledgements. This paper evolved from [AH07] which began during the author’s visit to Ben Gurion University in September 2006. The idea to use games in the iterated forcing construction of [AH07] was due to the author, but was undoubtedly influenced by the author’s readings of [She00a]. The iteration scenario in the proof of the consistency of $(\ast_s)_{\omega_1}$ with the existence of a nonspecial Aronszajn tree and CH is based on the scenario from [AH07] which was joint work resulting from discussions with Abraham during the visit. We thank Uri Abraham for patiently going through the details of Shelah’s theory on preserving nonspecialness with us, cf. [She98, Chapter IX], and for providing an atmosphere conducive to mathematical research during the visit.

2. PREREQUISITES

2.1. Terminology. For a model M and a poset $P \in M$, we write $\text{Gen}(M, P)$ for the collection of filters $G \subseteq P \cap M$ that are generic over M , i.e. $D \cap G \cap M \neq \emptyset$ for every dense $D \subseteq P$ in M . For $p \in P \cap M$, $\text{Gen}(M, P, p)$ denotes those $G \in \text{Gen}(M, P)$ with $p \in G$. A condition $q \in P$ is *generic over M* , also called *(M, P) -generic* if $q \Vdash \dot{G}_P \cap M \in \text{Gen}(M, P)$. We write $\text{gen}(M, P)$ for the set of (M, P) -generic conditions. As usual, a poset P is proper if for every countable elementary $M \prec H_\kappa$, for κ some sufficiently large regular cardinal, with $P \in M$, every $p \in P \cap M$ has an extension that is generic over M , i.e. $\text{gen}(M, P, p) \neq \emptyset$.

We let $\text{Gen}^+(M, P)$ denote all $G \in \text{Gen}(M, P)$ with a common extension in P , i.e. some $q \in P$ with $q \geq p$ for all $p \in G$; and $\text{Gen}^+(M, P, p)$ is defined similarly. Following [Abr06], we say that a condition is *completely (M, P) -generic* if $q \Vdash \dot{G}_P \cap M \in \text{Gen}^+(M, P)$. Note that for a separative poset P this is equivalent to $\{p \in M : q \geq p\} \in \text{Gen}^+(M, P)$. We write $\text{gen}^+(M, P)$ for the set of completely (M, P) -generic conditions, and we say that a poset P is *completely proper* if for every M as above, $\text{gen}^+(M, P, p) \neq \emptyset$ for every $p \in P \cap M$; or equivalently, $\text{Gen}^+(M, P, p) \neq \emptyset$. It is easily seen that P is completely proper iff it is proper and does not add new reals.

A *tower* of elementary submodels refers to a continuous \in -chain $M_0 \in M_1 \in \dots$ such that $\{M_\xi : \xi \leq \alpha\} \in M_{\alpha+1}$ for all α . For any collection \mathcal{M} , define $\text{gen}(\mathcal{M}, P, p) = \bigcap_{M \in \mathcal{M}} \text{gen}(M, P, p)$. For a tower \vec{M} of elementary submodels of some H_κ , we write $M_0 \prec M_1 \prec \dots$ to emphasize the fact that lower members are elementarily included in higher ones. A poset P is *α -proper* if every tower \vec{M} of countable height $\alpha+1$ consisting of countable elementary submodels of H_κ , with κ sufficiently large and regular and $P \in M_0$: every $p \in P \cap M_0$ has an extension generic over all members of the tower, i.e. $\text{gen}(\{M_\xi : \xi \leq \alpha\}, P, p) \neq \emptyset$. For a class \mathcal{E} of elementary submodels, we say that a poset P is *\mathcal{E} - α -proper* to indicate

that $\text{gen}(\{M_\xi : \xi \leq \alpha\}, P, p) \neq \emptyset$ for all $p \in P \cap M_0$, whenever $M_0 \prec \dots \prec M_\alpha$ is a tower of members of \mathcal{E} with $P \in M_0$.

For a countable model M , we let

$$\delta_M = \sup(\omega_1 \cap M). \quad (3)$$

When M is either an elementary submodel of H_{\aleph_1} or a transitive model then we have $\delta_M = \omega_1 \cap M$.

A *quasi order* is a pair (Q, \leq) where \leq is a reflexive and transitive relation on Q . A subset $C \subseteq Q$ is called *convex* if for all $p \leq q$ in C , $p \leq r \leq q$ implies $r \in C$.

For any binary relation $R \subseteq A \times B$ and $x \subseteq A$, we write $R[x]$ for the image of x under R , i.e. $R[x] = \{y \in B : R(x, y)\}$.

For any two sets x and y of ordinals, we write $x \sqsubseteq y$ to indicate that y *end extends* x , i.e. x is an *initial segment* of y .

For a tree T , we write T_α for the α^{th} level. An ω_1 -*tree* is a tree of height ω_1 with all levels countable. An *Aronszajn tree* is an ω_1 -tree with no cofinal (i.e. uncountable) branches. A tree is called *special* if it can be decomposed into countably many antichains. Note that a special ω_1 -tree must be Aronszajn. For $R \subseteq \omega_1$ we write $T \upharpoonright R$ for the restriction $\bigcup_{\alpha \in R} T_\alpha$ of T to levels in R . We write $\text{pred}_T(t) = \{u \in T : u <_T t\}$ for the set of predecessor of t .

We follow the standard set theoretic convention of writing V for the class of all sets.

2.2. Combinatorics. By an *ideal of sets* we of course mean an ideal \mathcal{I} in some lattice of sets, i.e. $\mathcal{I} \subseteq \mathcal{P}(S)$ for some fixed set S and \mathcal{I} is closed under subsets and pairwise unions of its members. We also say that \mathcal{I} is an *ideal on S* . For a cardinal λ , a λ -*ideal* is a λ -complete ideal, i.e. it is closed under unions of cardinality less than λ . A σ -*ideal* means an \aleph_1 -ideal, i.e. closed under countable unions.

Definition 2.1. Let (Q, \leq) be a quasi order. A subset $A \subseteq Q$ is *cofinal* in the quasi ordering if every $q \in Q$ has an $a \geq q$ in A . To every quasi order, we associate $\mathcal{J}(Q, \leq) \subseteq \mathcal{P}(Q)$ consisting of all noncofinal subsets of Q .

Lemma 2.2. Let (Q, \leq) be a quasi order. Then $\mathcal{J}(Q, \leq)$ is a lower set. Moreover:

- (a) If Q has no maximum elements then $Q \subseteq \mathcal{J}(Q, \leq)$, i.e. $\mathcal{J}(Q, \leq)$ contains all of the singletons of Q .
- (b) If $A \subseteq Q$ then $\mathcal{J}(A, \leq) \subseteq \mathcal{J}(Q, \leq)$.
- (c) If $I \subseteq Q$ is directed, then $\mathcal{J}(I, \leq)$ is a proper ideal on Q . More generally, if I is λ -directed then $\mathcal{J}(I, \leq)$ is a λ -ideal.

Proof of (c). Clearly Q is cofinal in itself. Thus $\mathcal{J}(I)$ is a proper ideal, i.e. $Q \notin \mathcal{J}(I)$. Suppose I is λ -directed, and $\mathcal{A} \subseteq \mathcal{J}(I)$ with $|\mathcal{A}| < \lambda$. For each $A \in \mathcal{A}$, there exists $i_A \in I$ with no element of A above it. By directedness, $\{i_A : A \in \mathcal{A}\}$ has an upper bound $j \in I$. Then j witnesses that $\bigcup \mathcal{A} \in \mathcal{J}(I)$. \square

We isolate the role of the property that a family of subsets of an ordinal θ , has no countable decomposition of θ into orthogonal pieces. This has already been done but in less generality in [Tod00].

Proposition 2.3. *Suppose \mathcal{F} is a family of subsets of some ordinal θ . Let $\alpha \leq \theta$ be the least ordinal with no countable decomposition into sets orthogonal to \mathcal{F} . Then α has uncountable cofinality.*

Lemma 2.4. *Let λ be a cardinal, and let \mathcal{F} be a directed subset of the quasi order $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a σ -ideal. Suppose the ordinal θ satisfies*

- (i) θ has no decomposition into (strictly) less than λ many sets orthogonal to \mathcal{F} ,
- (ii) every $\xi < \theta$ has a decomposition into less than λ many sets orthogonal to \mathcal{F} .

Then for every family \mathcal{X} of cofinal subsets of $(\mathcal{F}, \subseteq^)$ with $|\mathcal{X}| < \lambda$, and every $\xi < \theta$, there exists $\alpha \geq \xi$ in θ such that*

$$\{x \in X : \alpha \in x\} \text{ is cofinal in } (\mathcal{F}, \subseteq^*) \text{ for all } X \in \mathcal{X}. \quad (4)$$

Proof. Let \mathcal{F} be a directed subset of $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F})$ a σ -ideal.

Sublemma 2.4.1. *For every \subseteq^* -cofinal $X \subseteq \mathcal{F}$, if $Y \subseteq \theta$ and $\{x \in X : \alpha \in x\} \in \mathcal{J}(\mathcal{F})$ for all $\alpha \in Y$, then Y is orthogonal to \mathcal{F} .*

Proof. Suppose that some $Y \subseteq \theta$ is not orthogonal to \mathcal{F} , say $z \subseteq Y$ is infinite and $z \subseteq y$ for some $y \in \mathcal{F}$. Assume without loss of generality that z is countable. Since X is cofinal while \mathcal{F} is \subseteq^* -directed, $\{x \in X : z \subseteq^* x\}$ is cofinal. Therefore, it follows from the fact that z is countable while $\mathcal{J}(\mathcal{F})$ is a σ -ideal that there exists a finite $s \subseteq z$ with $\{x \in X : z \setminus s \subseteq x\}$ cofinal, completing the proof. \square

Now suppose θ and \mathcal{X} are as in the hypothesis. Assume towards a contradiction that for every $\alpha \geq \xi$, the set $\{x \in X_\alpha : \alpha \in x\}$ is noncofinal for some $X_\alpha \in \mathcal{X}$. For each $X \in \mathcal{X}$, put $Y_X = \{\alpha \geq \xi : X_\alpha = X\}$. Then by the sublemma, $\{Y_X : X \in \mathcal{X}\}$ is a decomposition of $\theta \setminus \xi$ into less than λ many sets orthogonal to \mathcal{F} . But by (ii), we have a decomposition of θ into less than λ many orthogonal sets, contradicting (i). \square

Now we isolate the role played by the stronger property that the family has no stationary orthogonal subset of θ .

Lemma 2.5. *Suppose $S \subseteq \theta$ is stationary. Let \mathcal{F} be a directed subfamily of $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a σ -ideal, and with no stationary subset of S orthogonal to \mathcal{F} . Then for every $M \prec H_{\theta^+}$ with $\mathcal{F}, S \in M$ and $|M| < \text{cof}(\theta)$, for every cofinal $X \subseteq \mathcal{F}$ in M , $\{x \in X : \sup(\theta \cap M) \in x\}$ is cofinal.*

Proof. Suppose $X \in M$ is a cofinal subset of $(\mathcal{F}, \subseteq^*)$. Let $Y \subseteq S$ be the set of all $\alpha < \theta$ such that $\{x \in X : \alpha \in x\} \in \mathcal{J}(\mathcal{F}, \subseteq^*)$. Then Y is orthogonal to \mathcal{F} , because sublemma 2.4.1 applies here as well. Thus by assumption it is not stationary. Therefore, as $\sup(\theta \cap M) \in \theta$ and $Y \in M$, we cannot have $\sup(\theta \cap M) \in Y$. \square

We verify here that principles such as $(*)$, $(*)_s$ and $(*)_c$ do not become weaker when the additional requirement that the families are P -ideals is imposed. Notice however, that this argument does not apply to the abstract game theoretic principles.

Lemma 2.6. *Let \mathcal{H} be a directed subfamily of $(\mathcal{P}(\omega_1), \subseteq^*)$. Supposing that X is a subset of ω_1 that is locally in the ideal $\langle \mathcal{H} \cup \text{Fin}(\omega_1) \rangle$ generated by $\mathcal{H} \cup \text{Fin}(\omega_1)$, there exists a finite $s \subseteq X$ such that $X \setminus s$ is locally in $\downarrow \mathcal{H}$.*

Proof. For each $\delta < \omega_1$, $X \cap \delta \in \langle \mathcal{H} \cup \text{Fin}(\omega_1) \rangle$. Hence there exists a finite $\mathcal{F}_\delta \subseteq \mathcal{H}$ and a finite $s_\delta \subseteq \omega_1$ such that $X \cap \delta \subseteq \bigcup \mathcal{F}_\delta \cup s_\delta$. Since \mathcal{H} is directed, \mathcal{F}_δ has a \subseteq^* -bound $y_\delta \in \mathcal{H}$. Hence there is a finite $t_\delta \subseteq \omega_1$ such that $\bigcup \mathcal{F}_\delta \subseteq y_\delta \cup t_\delta$. Thus

$$X \cap \delta \setminus (s_\delta \cup t_\delta) \subseteq y_\delta. \quad (5)$$

Pressing down, there exists a stationary $S \subseteq \omega_1$ such that $s_\delta = s$ and $t_\delta = t$ for all $\delta \in S$. It now follows that $X \setminus (s \cup t)$ is locally in $\downarrow \mathcal{H}$. \square

Remark 2.7. We only dealt with the case $\theta = \omega_1$. For general θ of uncountable cofinality the same argument applies, so long as $\text{otp}(X) = \omega_1$. This is of no concern for the principle $(*)$, but for $(*)_s$ and $(*)_c$, where we want a closed uncountable set, we need at least one limit of uncountable cofinality relative to X . For example we could have $\text{otp}(X) = \omega_1 + 1$, but the top point might prevent X from being locally in $\downarrow \mathcal{H}$.

Nevertheless, if we weaken the principles $(*)_s$ or $(*)_c$ so that X is only required to be closed at limits of countable cofinality relative to X (and thus X can have order type ω_1), we can prove that they are still equivalent to the principles stated in section 1.1. This is left as an exercise for the interested reader.

2.3. Game Theory. We shall follow the convention that the first move of any game is move 0, and that the k^{th} move refers to move k . **Caution.** There is some potential for confusion, because this means for example that the 1st move is move 1, whereas the 0th move is really the ‘first’ move.

In the simplest n player game, there is one winner and all of the other players lose, and there are exactly n possible outcomes. More generally, the game can result in any possible ranking (not necessarily injective) of the n players for a total of

$$\sum_{i=0}^{n-1} p(n, n-i) \cdot (n-i)! \quad (6)$$

possible outcomes, where $p(n, k)$ is the number of partitions of n into k pieces. The first few terms are $n!$, $\binom{n}{2}(n-1)!$, $\left[\binom{n}{3} + 3\binom{n}{4}\right](n-2)!$, $\left[\binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6}\right](n-3)!$, \dots . We say that player X *wins* if it is ranked strictly above all of the other players. We say that it *loses* if there is some other player ranked strictly above him. Accordingly we distinguish between a *winning strategy* for player X and a *nonlosing strategy* for X . In the case of a two player game, there are $p(2, 2) \cdot 2! + p(2, 1) \cdot 1! = 2 + 1 = 3$ possible outcomes, either player can win and there can be a tie, i.e. draw, where neither player wins or loses.

The following concept is useful for dealing with completeness systems.

Definition 2.8. Let \mathcal{D} be a game of length δ and X a player of \mathcal{D} . A *forward strategy* for X in the game \mathcal{D} is a strategy Φ for X such that for any position P in the game \mathcal{D} , where the game has not yet ended and it is X 's turn to play, $\Phi(P)$ gives a move for X . A *forward winning* [nonlosing] *strategy* for X is a forward strategy Φ for X such that X wins [does not lose] the game so long as there exists $\xi < \delta$ such that X plays according to Φ on move η for all $\eta \geq \xi$.

Thus the point of a forward strategy for X is that X can decide at any point in the game to start playing according to the strategy. Of course a forward (winning [nonlosing]) strategy is in particular a (winning [nonlosing]) strategy.

Definition 2.9. Let $\mathcal{D}(M, x, a_0, \dots, a_{n-1})$ be a parameterized game with a fixed number of players n (with respect to the parameters). Suppose that the first parameter M is taken from some family \mathcal{S} , the second parameter is a subset of M and that the third parameter $\vec{a} = a_0, \dots, a_{n-1}$ is taken from some family \mathcal{T} .

We consider a function F with domain \mathcal{S} where

$$F(M) : \mathcal{T} \cap M \rightarrow \mathcal{P}(\mathcal{P}(M)) \quad \text{for all } M \in \mathcal{S}, \quad (7)$$

i.e. $F(M)(\vec{a}) \subseteq \mathcal{P}(M)$ for all $\vec{a} \in \mathcal{T} \cap M$. We think of F as describing a notion of suitability over M for the second parameter. It is required to satisfy

$$F(M)(\vec{a}) \neq \emptyset \quad \text{for all } M \in \mathcal{S} \text{ and all } \vec{a} \in \mathcal{T} \cap M. \quad (8)$$

Suppose X is some player in the parameterized game. For any $\mathcal{E} \subseteq \mathcal{S}$, and any property Φ where Φ has $n+3$ variables, we say that Φ holds for X , \mathcal{E} - F -globally, if $\Phi(X, M, x, \vec{a})$ holds whenever

- (i) $M \in \mathcal{E}$,
- (ii) $x \in F(M)(\vec{a})$,
- (iii) $\vec{a} \in \mathcal{T} \cap M$.

Particularly, X has a winning [nonlosing] strategy in the game \mathcal{D} , \mathcal{E} - F -globally, means that X has a winning [nonlosing] strategy in the game $\mathcal{D}(M, x, \vec{a})$ whenever M , x and \vec{a} satisfy (i)–(iii).

When we say that a property holds F -globally, we mean \mathcal{S} - F -globally.

We will consider parameterized games that are given by some first order definition in the language of set theory. In some situations, e.g. in moving between forcing extensions, it is important to distinguish between a parameterized game as a mathematical object or a defined notion.³ The following notation is used to help accomplish this.

Notation 2.10. Typically, when dealing with the parameterized game as a defined notion, we also want the suitability function F to be a defined notion, and vice versa. Thus we write a formula in place of F to indicate that the parameterized game, the function F (and thus \mathcal{S}) and \mathcal{T} are first order definable with no parameters. We may display objects to indicate that a particular variable is fixed, and also when we want to specify some part as an object (e.g. \mathcal{E} in

³ As it turned out, this never became an issue in the present article.

example 2.12). In the case when some of the parameters b_0, \dots, b_{m-1} are fixed, e.g. some property holds globally for player X in the game $\mathcal{D}(b_0, \dots, b_{m-1})$ (see e.g. example 2.12), equation (8) is only required for parameters $\vec{a} \in \mathcal{T}$ such that $\vec{a} = b_0, \dots, b_{m-1}, a_m, \dots, a_{n-1}$ and for $M \ni \vec{a}$.

Example 2.11. Let ψ be a first order formula with $n + 2$ free variables. When we say that a property Φ of the parameterized game \mathcal{D} holds ψ -globally for X we are indicating that the parameterized game \mathcal{D} as well as \mathcal{S} and \mathcal{T} are definable by first order formulae with no additional parameters and that the function F is given by

$$F(M)(\vec{a}) = \{x \subseteq M : \psi(M, x, \vec{a})\}. \quad (9)$$

Notice that equation (8) becomes

$$\varphi_{\mathcal{S}}(M) \wedge \varphi_{\mathcal{T}}(\vec{a}) \rightarrow \exists x \subseteq M \psi(M, x, \vec{a}) \quad \text{for all } M \text{ and all } \vec{a} \in M. \quad (10)$$

Example 2.12. Letting $n = 1$, suppose ψ is a formula with 3 free variables. Suppose \mathcal{E} and b are sets. Then saying that X has a winning strategy in the game $\mathcal{D}(b)$, \mathcal{E} - ψ -globally, indicates that the parameterized game $\mathcal{D}(M, x, a)$ can be identified with a formula, but only indicates that X has a winning strategy when $a = b$. It also indicates that F and \mathcal{T} are definable with no parameters, but is specifying the object \mathcal{E} .

Definition 2.13. For two first order formulae $\varphi(v_0, \dots, v_n)$ and $\psi(v_0, \dots, v_n)$, we write $\varphi \rightarrow \psi$ for its universal closure. We say that *provably* $\varphi \rightarrow \psi$ to indicate that

$$\text{ZFC} \vdash \ulcorner \forall v_0, \dots, v_n \varphi(v_0, \dots, v_n) \rightarrow \psi(v_0, \dots, v_n) \urcorner. \quad (11)$$

Let us point out an obvious relationship.

Proposition 2.14. *Suppose $\varphi \rightarrow \psi$. Then if X has a [forward] (winning) strategy for X , \mathcal{E} - ψ -globally in a parameterized game \mathcal{D} , then X also has a [forward] (winning) strategy \mathcal{E} - φ -globally in \mathcal{D} .*

Definition 2.15. Let \mathcal{D} be some game of length δ , with player A playing first. For a position P in the game \mathcal{D} with A to play, we define the restricted game $\mathcal{D} \upharpoonright P$ as a game with the same players, playing in the same order, of length $\delta - |P|$. The rules of the game are that in position Q of $\mathcal{D} \upharpoonright P$ with X to play, m is a valid move for X if m is a valid move for X in the position $P \frown Q$ of the game \mathcal{D} . And if Q is the position at the end of the game $\mathcal{D} \upharpoonright P$, then player X wins [loses] in the play Q if X wins [loses] the play $P \frown Q$ of the game \mathcal{D} .

The following lemma is used to obtain forward strategies.

Lemma 2.16. *Assume that $\mathcal{D}(M, x, \vec{a})$ is a parameterized game with a fixed number of players where player A always moves first, and a fixed length δ with δ indecomposable, with respect to the parameters. Suppose that player X has a winning [nonlosing] strategy in the parameterized game \mathcal{D} , \mathcal{E} - F -globally. If every suitable triple (M, x, \vec{a}) and every position P of the game $\mathcal{D}(M, x, \vec{a})$ with A to play, has an $x' \subseteq M$ and $\vec{a}' \in \mathcal{T} \cap M$ such that*

- (a) $x' \in F(M)(\vec{a}')$,
- (b) $\mathcal{D}(M, x', \vec{a}') = \mathcal{D}(M, x, \vec{a}) \upharpoonright P$,

then X has a forward winning [nonlosing] strategy in \mathcal{D} , \mathcal{E} - F -globally.

Proof. We use the following notation in this proof.

Notation 2.17. If P is a position in some game \mathcal{D} , where A moves first, and it is player X 's turn to make its ξ^{th} move (note that this entails $|P|$ is either ξ or $\xi + 1$ depending on whether or not $X = A$, resp.), then for each $\gamma \leq \xi$, we let $P_\gamma \sqsubseteq P$ be the position preceding P where it is A 's turn to make its γ^{th} move.

Assuming the hypotheses, take a suitable triple (M, x, \vec{a}) , and let $\Phi(M, x, \vec{a})$ be a winning [nonlosing] strategy for X in the game $\mathcal{D}(M, x, \vec{a})$. We define a forward strategy $\Phi'(M, x, \vec{a})$ for X as follows. Take a position P in the game $\mathcal{D}(M, x, \vec{a})$ where it is X 's turn to make its ξ^{th} move. For each $\gamma \leq \xi$, let $x_{P,\gamma}$ and $\vec{a}_{P,\gamma}$ be the x' and \vec{a}' guaranteed by the hypothesis with $P := P_\gamma$. Then by (a) and (b), we can let γ_P be the least ordinal $\gamma \leq \xi$ such that $P = P_\gamma \frown Q^\gamma$ where Q^γ is the result of X playing according to the strategy $\Phi(M, x_{P,\gamma}, \vec{a}_{P,\gamma})$ in the game $\mathcal{D}(M, x_{P,\gamma}, \vec{a}_{P,\gamma})$. And then

$$\Phi'(M, x, \vec{a})(P) = \Phi(M, x_{P,\gamma_P}, \vec{a}_{P,\gamma_P})(Q^{\gamma_P}) \quad (12)$$

defines a strategy for X in the game $\mathcal{D}(M, x, \vec{a})$ by (b), which is moreover a forward strategy by its definition.

Suppose that the game $\mathcal{D}(M, x, \vec{a})$ has been played, where X has played according to $\Phi'(M, x, \vec{a})$ on every move $\alpha \geq \xi$. Assuming that ξ is the least such ordinal, then by equation (12), from its ξ^{th} move on, X has played according to the strategy $\Phi(M, x_{P,\xi}, \vec{a}_{P,\xi})$ in the game $\mathcal{D}(M, x_{P,\xi}, \vec{a}_{P,\xi})$, where P is the position in the game $\mathcal{D}(M, x, \vec{a})$ when it was X 's turn to make its ξ^{th} move. Therefore, X wins [does not lose] in the game $\mathcal{D}(M, x_{P,\xi}, \vec{a}_{P,\xi})$, and thus X wins [does not lose] the game $\mathcal{D}(M, x, \vec{a})$ by (b). \square

Remark 2.18. The only role of indecomposability is that it is entailed by (b).

3. GENERAL RESULTS

3.1. The forcing notion.

Definition 3.1. For two families \mathcal{F} and \mathcal{H} of subsets of some ordinal, let $\mathcal{R}(\mathcal{F}, \mathcal{H})$ be the poset consisting of all pairs $p = (x_p, \mathcal{X}_p)$ where

- (i) $x_p \in \mathcal{H}$,
- (ii) \mathcal{X}_p is a nonempty countable family of cofinal subsets of $(\mathcal{F}, \subseteq^*)$,
- (iii) $\{y \in X : x_p \subseteq y\}$ is cofinal in $(\mathcal{F}, \subseteq^*)$ for all $X \in \mathcal{X}_p$,

ordered by q extends p if

- (iv) $x_q \supseteq x_p$ (i.e. x_q end extends x_p with respect to the ordinal ordering),
- (v) $\mathcal{X}_q \supseteq \mathcal{X}_p$.

We write $\mathcal{R}(\mathcal{H})$ for $\mathcal{R}(\mathcal{H}, \mathcal{H})$.

Remark 3.2. Note we can assume without loss of generality that $\mathcal{H} \subseteq \partial(\mathcal{F})$, i.e.

$$\mathcal{R}(\mathcal{F}, \mathcal{H}) = \mathcal{R}(\mathcal{F}, \mathcal{H} \cap \partial(\mathcal{F})), \quad (13)$$

where $\partial(\mathcal{F})$ is the set of all x such that $\{y \in \mathcal{F} : x \subseteq y\}$ is cofinal in $(\mathcal{F}, \subseteq^*)$. Obviously $\partial(\mathcal{F}) \subseteq \downarrow \mathcal{F}$.

Proposition 3.3. *Let $(\mathcal{F}, \subseteq^*)$ be λ -directed and \mathcal{H} arbitrary. Then every $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ and every $A \in [\mathcal{F}]^{<\lambda}$ has a $y \in \mathcal{F}$ such that $x_p \subseteq y$ and y is a \subseteq^* -upper bound of A .*

Proof. The “nonempty” in (ii) is needed here. Take any $X \in \mathcal{X}_p$. There exists an upper bound $y' \in \mathcal{F}$ of A because \mathcal{F} is λ -directed, and thus by (iii), there exists $y \supseteq^* y'$ with $x_p \subseteq y$ in X , as required. \square

Definition 3.4. Let \mathcal{F} and \mathcal{H} be families of subsets of some ordinal θ . We say that \mathcal{H} is \mathcal{F} -*extendable* if for every $x \in \mathcal{H}$, every countable family \mathcal{X} of \subseteq^* -cofinal subsets of \mathcal{F} such that $x \subseteq z$ for all $z \in X$ for all $X \in \mathcal{X}$, and every $\xi < \theta$, there exists $y \subseteq \theta$ such that

- (i) $y \supseteq x$
- (ii) $y \in \mathcal{H}$,
- (iii) $y \setminus \xi \neq \emptyset$,
- (iv) $\{z \in X : y \subseteq z\}$ is cofinal in \mathcal{F} for all $X \in \mathcal{X}$.

We just say that \mathcal{H} is *extendable* to indicate that it is \mathcal{H} -extendable.

Remark 3.5. The definition of \mathcal{H} being \mathcal{F} -extendable actually depends on the choice of θ . We shall always implicitly assume that θ can be computed from \mathcal{F} , i.e. it is the supremum of the ordinals appearing in \mathcal{F} , i.e. $\theta = \sup \bigcup \mathcal{F}$.

The preceding definition was tailored for the following density result that is crucial for obtaining the desired uncountable $X \subseteq \theta$.

Proposition 3.6. *Let \mathcal{F} and \mathcal{H} be families of subsets of θ . Suppose that \mathcal{H} is \mathcal{F} -extendable. Then for every $\xi < \theta$,*

$$\mathcal{D}_\xi = \{p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) : \sup(x_p) \geq \xi\} \text{ is dense.} \quad (14)$$

For a filter $G \subseteq \mathcal{R}(\mathcal{F}, \mathcal{H})$, the generic object is

$$X_G = \bigcup_{p \in G} x_p \subseteq \theta. \quad (15)$$

We make the obvious observations.

Proposition 3.7. *X_G is the union of a chain in $(\mathcal{H}, \sqsubseteq)$.*

Proposition 3.8. *Every proper initial segment $y \sqsubset X_G$ has an $x \sqsupseteq y$ in \mathcal{H} . I.e. every proper initial segment of X_G is in $\downarrow(\mathcal{H}, \sqsubseteq)$.*

3.2. The associated games. Following is the natural game associated with our forcing notion.

Definition 3.9. Let \mathcal{F} and \mathcal{H} be families of subsets of an ordinal θ , $y \subseteq \theta$ and $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ with $x_p \subseteq y$. Define the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ with players *Extender* and *Complete* of length ω . Extender plays first and on move 0 must play p_0 so that

- p_0 extends p .

On the k^{th} move:

- Extender plays $p_k \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ satisfying
 - (1) p_k extends p_i for all $i = 0, \dots, k-1$,
 - (2) $x_{p_k} \subseteq y \setminus \bigcup_{i=0}^{k-1} s_i$,
- Complete plays a finite $s_k \subseteq y \setminus x_{p_k}$.

Complete wins if the sequence p_k ($k < \omega$) has a common extension in $\mathcal{R}(\mathcal{F}, \mathcal{H})$, and Extender wins otherwise.

Notation 3.10. For a centered subset C of some poset P , we let $\langle C \rangle$ denote the filter on P generated by C .

Definition 3.11. We define a variation of the game $\mathfrak{D}_{\text{cmp}}$ as follows. For some M , for families $\mathcal{F}, \mathcal{H} \in M$ of subsets of some ordinal, $y \subseteq M$ and $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$ with $x_p \subseteq y$, we define the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$. It has the same rules, but with the additional rule that Extender's k^{th} move, p_k or (p_k, X_k) , respectively, must satisfy

$$p_k \in M. \tag{16}$$

This game has three possible outcomes, determined as follows:

- (i) Extender loses (i.e. Complete wins) if $\langle p_k : k < \omega \rangle \notin \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$,
- (ii) the game is drawn (i.e. a tie) if $\langle p_k : k < \omega \rangle \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$,
- (iii) Extender wins the game if $\langle p_k : k < \omega \rangle \in \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$ but (ii) fails.

The game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$ is especially interesting for us because a draw in this game corresponds precisely with complete genericity.

Proposition 3.12. *Let p_k denote Extender's k^{th} move in the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$. Then the game results in a draw iff $\langle p_k : k < \omega \rangle \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$.*

The following augmented game is used for preserving the nonspecialness of trees.

Definition 3.13. We define an augmented game $\mathfrak{D}_{\text{gen}}^*(M, y, \mathcal{F}, \mathcal{H}, p)$. Again Extender moves first with $p_0 \geq p$ in M , but Extender additionally plays $X_k \subseteq \mathcal{F}$ on each move. The whole point is that X_k is *not* required to be in M . On the k^{th} move:

- ◆ Extender plays (p_k, X_k) where $p_k \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$ and $X_k \subseteq \mathcal{F}$ satisfy
 - (1) p_k extends p_i for all $i = 0, \dots, k-1$,
 - (2) $x_{p_k} \subseteq y \setminus \bigcup_{i=0}^{k-1} s_i$,
 - (3) $\{x \in X_i : x_{p_k} \subseteq x\}$ is cofinal in $(\mathcal{F}, \subseteq^*)$ for all $i = 0, \dots, k$,
- ◆ Complete plays a finite $s_k \subseteq y \setminus x_{p_k}$.

The possible outcomes are:

- (i) Complete wins if $\langle p_k : k < \omega \rangle \notin \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$.
- (ii) The game is drawn if $\langle p_k : k < \omega \rangle \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$ and moreover $\{p_k : k < \omega\}$ has a common extension $q \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ with
 - ◆ $\{X_k : k < \omega\} \subseteq \mathcal{X}_q$,
- (iii) Complete loses the game if $\langle p_k : k < \omega \rangle \in \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$ and (ii) fails.

Proposition 3.14. *At the end of any of the three games $\mathfrak{D}_{\text{cmp}}$, $\mathfrak{D}_{\text{gen}}$ or $\mathfrak{D}_{\text{gen}}^*$,*

$$\bigcup_{k < \omega} x_{p_k} \subseteq y \setminus \bigcup_{k < \omega} s_k. \quad (17)$$

The augmented game $\mathfrak{D}_{\text{gen}}^*$ ‘includes’ the game $\mathfrak{D}_{\text{gen}}$ in the following sense.

Proposition 3.15. *If $((p_0, X_0), s_0), \dots, ((p_k, X_k), s_k)$ is a position in the game $\mathfrak{D}_{\text{gen}}^*(y, \mathcal{F}, \mathcal{H}, p)$ then $(p_0, s_0), \dots, (p_k, s_k)$ is a position in the game $\mathfrak{D}_{\text{gen}}(y, \mathcal{F}, \mathcal{H}, p)$. Conversely, if $(p_0, s_0), \dots, (p_k, s_k)$ is a position in the game $\mathfrak{D}_{\text{gen}}(y, \mathcal{F}, \mathcal{H}, p)$ then $((p_0, \mathcal{F}), s_0), \dots, ((p_k, \mathcal{F}), s_k)$ is a position in the game $\mathfrak{D}_{\text{gen}}^*(y, \mathcal{F}, \mathcal{H}, p)$.*

Proof. $\mathcal{X}_p \neq \emptyset$ in definition 3.1(ii) is used for the converse. \square

Proposition 3.16. *A nonlosing strategy for Complete in the game $\mathfrak{D}_{\text{gen}}^*(M, y, \mathcal{F}, \mathcal{H}, p)$ yields a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$.*

Proof. At a position $(p_0, s_0), \dots, p_k$ (with Complete to move) in the game $\mathfrak{D}_{\text{gen}}(y, \mathcal{F}, \mathcal{H}, p)$, by proposition 3.15, $((p_0, \mathcal{F}), s_0), \dots, (p_k, \mathcal{F})$ is a position in the game $\mathfrak{D}_{\text{gen}}^*(y, \mathcal{F}, \mathcal{H}, p)$. Thus the strategy for Complete in the game $\mathfrak{D}_{\text{gen}}^*(y, \mathcal{F}, \mathcal{H}, p)$ gives a move s_k . This defines a nonlosing strategy for Complete in the game $\mathfrak{D}_{\text{gen}}(y, \mathcal{F}, \mathcal{H}, p)$, because a draw or play of the game $\mathfrak{D}_{\text{gen}}^*(y, \mathcal{F}, \mathcal{H}, p)$ results in draw in the game $\mathfrak{D}_{\text{gen}}(y, \mathcal{F}, \mathcal{H}, p)$. \square

We relate the ‘completeness’ game to the latter ‘genericity’ game.

Proposition 3.17. *A (forward) winning strategy for Complete in the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ yields a (forward) nonlosing strategy for Complete in the game $\mathfrak{D}_{\text{gen}}^*(M, y, \mathcal{F}, \mathcal{H}, p)$.*

Proof. At a position $((p_0, X_0), s_0), \dots, (p_k, X_k)$ in the game $\mathfrak{D}_{\text{gen}}^*(M, y, \mathcal{F}, \mathcal{H}, p)$, let $\bar{p}_i = p_i \cup \bigcup_{j=0}^i X_j$ for each $i = 0, \dots, k$. Then each $\bar{p}_i \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ by rule (3), and thus $(\bar{p}_0, s_0), \dots, \bar{p}_k$ is a position in the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$. Thus if s_k is played according to Complete’s winning strategy in the latter game, then Complete wins the latter game. This means that $\{\bar{p}_0, \bar{p}_1, \dots\}$ has a common extension q , and then q extends $\{p_0, p_1, \dots\}$ and satisfies $\{X_0, X_1, \dots\} \subseteq \mathcal{X}_q$, yielding a draw in the former game. \square

Proposition 3.18. *Assume \mathcal{F} and \mathcal{H} are families of subsets of θ , $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ and $x_p \subseteq y$. Let $t \in \text{Fin}(\theta \setminus x_p)$. Then Complete has a winning strategy in the game $\mathcal{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ iff it has a winning strategy in the game $\mathcal{D}_{\text{cmp}}(y \cup t, \mathcal{F}, \mathcal{H}, p)$. Similarly for the games \mathcal{D}_{gen} and $\mathcal{D}_{\text{gen}}^*$.*

Proof. Assume Complete has a winning strategy in $\mathcal{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$. Assume without loss of generality that $y \cap t = \emptyset$. Then a winning strategy for Complete in $\mathcal{D}_{\text{cmp}}(y \cup t, \mathcal{F}, \mathcal{H}, p)$ is given by playing $s_k \cup t$ on move k where s_k is played according to the strategy for the former game. This is because they both give identical restrictions on Extender's choice of moves according to rule (2). Conversely, if Complete plays $s'_k \setminus t$ in the former game, where s'_k has been played in the latter game, then Extender has less freedom to move in the former game. \square

We are interested in making finite extensions of the third parameter when dealing with completeness systems; but unfortunately, the above approach does not seem to generalize to forward winning strategies.

The argumentation in the preceding proof, i.e. the fact that restricting Extender's moves is favourable for Complete in any of the 3 games, does show the following.

Proposition 3.19. *Let Φ be a (forward) winning, resp. nonlosing, strategy for Complete in the game $\mathcal{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ or the game $\mathcal{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$, respectively. Then Complete wins, resp. does not lose, the game whenever it plays $s_k \supseteq \Phi(P_k)$ on every move (after some point) in the game, where P_k is the position after Extender makes its k^{th} move. Similarly, for the augmented game $\mathcal{D}_{\text{gen}}^*$.*

Next we isolate the role played by the \subseteq^* -cofinal subsets of the family \mathcal{F} .

Remark 3.20. Henceforth, when we write H_κ there is a tacit assumption that κ is a sufficiently large regular cardinal for the argument at hand. This will always be in the context of some pair $(\mathcal{F}, \mathcal{H})$ of subfamilies of $\mathcal{P}(\theta)$. It will always be sufficiently large as long as $\mathcal{R}(\mathcal{F}, \mathcal{H}) \in H_\kappa$, e.g. when $\kappa \geq (2^{\max\{|\mathcal{F}|, |\mathcal{H}|\}})^+$.

Definition 3.21. We say that a family \mathcal{F} of sets is *closed under finite reductions* to indicate that it is closed under finite set subtraction, i.e. $x \setminus s \in \mathcal{F}$ for all $x \in \mathcal{F}$ and all finite $s \subseteq x$.

Lemma 3.22. *Let \mathcal{F} and \mathcal{H} be subsets of $\mathcal{P}(\theta)$, with \mathcal{F} closed under finite reductions, and let $M \prec H_\kappa$ be a model containing \mathcal{F} and \mathcal{H} satisfying*

$$x \subseteq M \quad \text{for all } x \in \mathcal{F} \cap M. \quad (18)$$

Suppose $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$, let $Q \subseteq \mathcal{R}(\mathcal{F}, \mathcal{H})$ be an element of M and assume that $y \subseteq \theta$ is a \subseteq^ -upper bound of $\mathcal{F} \cap M$ with $x_p \subseteq y$. Then one of the following two alternatives must hold.*

- (a) *There exists an extension q of p in $Q \cap M$ with $x_q \subseteq y$.*
- (b) *There exists a \subseteq^* -cofinal $X \subseteq \mathcal{F}$ such that*
 - (1) $x_p \subseteq x$ for all $x \in X$,
 - (2) *for no extension $q \in Q$ of p does there exist $z \in X$ satisfying $x_q \subseteq z$.*

Proof. Define Y to be the set of all $x \in \mathcal{F}$ for which there is some $y_x \supseteq^* x$ in \mathcal{F} , with $x_p \subseteq y_x$, such that no $q \geq p$ in Q satisfies $x_q \subseteq y_x$. Clearly $Y \in M$. Take $x \in \mathcal{F} \cap M$. Taking any $Z \in \mathcal{X}_p$, by elementarity there exists $z \in Z \cap M$ such that $x_p \subseteq z$ and $x \subseteq^* z$. Since $y \supseteq^* z$,

$$y_x = y \cap z \supseteq^* x. \quad (19)$$

And y_x is the result of removing a finite subset from z . Thus $y_x \in M$ as $z \subseteq M$, and $y_x \in \mathcal{F}$ by the assumption on \mathcal{F} .

Assume that alternative (a) fails. Then as $y_x \subseteq y$ is in M , by elementarity, y_x witnesses that $x \in Y$. Therefore, $Y = \mathcal{F}$ by elementarity, and thus $X = \{y_x : x \in \mathcal{F}\}$ is \subseteq^* -cofinal by (19). And since the y_x 's are witnesses, alternative (b) holds for X . \square

Corollary 3.22.1. *Let $\mathcal{F}, \mathcal{H}, M, p, Q$ and y all be as specified in lemma 3.22. Assume that $k+1$ moves have been made in either the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$, or the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$ with $y \subseteq M$, with Extender playing p_i on its i^{th} move, and that each $p_i \in M$ (in the former game). Then one of the following two alternatives must hold.*

- (a) *Extender has a move with $p_{k+1} \in Q \cap M$,*
- (b) *There exists a \subseteq^* -cofinal $X \subseteq \mathcal{F}$ such that*
 - (1) $x_{p_k} \subseteq x$ for all $x \in X$,
 - (2) *for no extension $q \in Q$ of p_k does there exist $z \in X$ satisfying $x_q \subseteq z$.*

Similarly for the augmented game $\mathfrak{D}_{\text{gen}}^(M, y, \mathcal{F}, \mathcal{H}, p)$.*

Proof. Let s_0, \dots, s_k denote the moves played so far by Complete. Lemma 3.22 is applied with $p := p_k$ and $y := y \setminus \bigcup_{i=0}^k s_i$. The second alternatives are identical, and thus if (b) fails, then there is an extension $p_{k+1} \geq p_k$ in $Q \cap M$ with $x_{p_{k+1}} \subseteq y \setminus \bigcup_{i=0}^k s_i$. Thus p_{k+1} satisfies the requirement (2) of the game, as needed. \square

The main purpose the side condition \mathcal{X}_p is to allow Extender to play inside a given dense subset of $\mathcal{R}(\mathcal{F}, \mathcal{H})$ in M .

Corollary 3.22.2. *In the situation of corollary 3.22.1, if Q is dense then Extender can always play $p_{k+1} \in Q \cap M$.*

Proof. Supposing towards a contradiction that Extender has no move with $p_{k+1} \in Q \cap M$, by corollary 3.22.1, there is a cofinal $X \subseteq \mathcal{F}$ as in alternative (b). Then $\bar{q} = (x_{p_k}, \mathcal{X}_{p_k} \cup \{X\})$ is a condition of $\mathcal{R}(\mathcal{F}, \mathcal{H})$ by (b1), and there exists $q \geq \bar{q}$ in Q by density. But $x_q \subseteq z$ for cofinally many $z \in X$ contradicting (b2). \square

Corollary 3.22.3. *Let $\mathcal{F}, \mathcal{H}, M, p, Q$ and y be as in lemma 3.22, with moreover M countable and $y \subseteq M$. Then Extender has nonlosing strategies in both of the games $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$ and $\mathfrak{D}_{\text{gen}}^*(M, y, \mathcal{F}, \mathcal{H}, p)$.*

Proof. Let $(D_k : k < \omega)$ enumerate all of the dense subsets of $\mathcal{R}(\mathcal{F}, \mathcal{H})$ in M . By corollary 3.22.2, Extender can always make move k with

$$p_k \in D_k \cap M. \quad (20)$$

This describes a nonlosing strategy, because at the end of the game, $\langle p_k : k < \omega \rangle \in \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$. \square

Definition 3.23. Let $\psi_{\min}(M, y, \mathcal{F}, \mathcal{H}, p)$ be a formula expressing the conjunction of

- (i) $x_p \subseteq y$,
- (ii) y is an upper bound of $(\mathcal{F} \cap M, \subseteq^*)$.

Later on we will use the fact that $\psi_{\min}(M, \cdot, \mathcal{F}, \mathcal{H}, p)$ defines a set that is second order definable over M .

Definition 3.24. Let $\phi_{\min}(y; \mathcal{F}, \mathcal{H}, p)$ be a second order formula expressing the conjunction of

- (i) $x_p \subseteq y$; formally, $\forall \alpha \in x_p \ y(\alpha)$,
- (ii) y is an upper bound of $(\mathcal{F}, \subseteq^*)$; formally, $\forall x \in \mathcal{F} \ y(\alpha)$ for all but finitely many $\alpha \in x$.

Proposition 3.25. Suppose M is a model of enough of $\text{ZFC} - \text{P}^4$ and $x \subseteq M$ for all $x \in \mathcal{F}$. Then for all $\mathcal{F}, \mathcal{H}, p \in M$ and all $y \subseteq M$, $\psi_{\min}(M, y, \mathcal{F}, \mathcal{H}, p) \leftrightarrow M \models \phi_{\min}(y; \mathcal{F}, \mathcal{H}, p)$.

Definition 3.26. All three of the games considered, \mathcal{D}_{cmp} , \mathcal{D}_{gen} and $\mathcal{D}_{\text{gen}}^*$, are viewed as parameterized games of the form $\mathcal{D}(M, x, a_0, a_1, a_2, a_3)$, as in definition 2.9, where a_3 is a “dummy” variable whose purpose is explained below. For example, $\mathcal{D}_{\text{cmp}}(M, x, a_0, a_1, a_2, a_3) \equiv \mathcal{D}_{\text{cmp}}(x, a_0, a_1, a_2)$ and $\mathcal{D}_{\text{gen}}(M, x, a_0, a_1, a_2, a_3) \equiv \mathcal{D}_{\text{gen}}(M, x, a_0, a_1, a_2)$. Define \mathcal{S} to be the class of all countable elementary submodels $M \prec H_\kappa$, with κ a regular uncountable cardinal. For a given cardinal κ , we define $\mathcal{S}_\kappa \subseteq \mathcal{S}$ by $\mathcal{S}_\kappa = \bigcup_{\mu \geq \kappa \text{ is regular}} \{M \prec H_\mu : |M| = \aleph_0\}$. \mathcal{T} is defined by $\varphi_{\mathcal{T}}(\mathcal{F}, \mathcal{H}, p, a_3)$ stating that \mathcal{F} and \mathcal{H} are families of sets of ordinals, $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ and $a_3 = \mathcal{R}(\mathcal{F}, \mathcal{H})^\top$, with the provision that we may restrict \mathcal{T} further when needed. Note that these games, as well as \mathcal{S} and \mathcal{T} , are definable without any additional parameters. In this setting, we use a formula ψ to describe the suitability function F ; we suppress the last free variable in ψ since a_3 obviously plays no role in the definition of F . The role played by a_3 , is that for any $M \in \mathcal{S}$, when $\vec{a} = (\mathcal{F}, \mathcal{H}, p, a_3) \in \mathcal{T} \cap M$ this implies that $\mathcal{R}(\mathcal{F}, \mathcal{H}) \in M$ and thus $M \prec H_\kappa$ for some sufficiently large cardinal κ as in remark 3.20. We could also use \mathcal{S}_κ below instead of \mathcal{S} , just as well.

For $\mathcal{E} \subseteq \mathcal{S}$, $\mathcal{E}_\kappa = \mathcal{E} \cap \mathcal{S}_\kappa$. Moreover, for $R \subseteq \theta$, we let

$$\mathcal{E}(R, \theta) = \{M \in \mathcal{S} : \sup(\theta \cap M) \in R\}, \quad (21)$$

and for $R \subseteq \omega_1$ we write $\mathcal{E}(R)$ for $\mathcal{E}(R, \omega_1)$. Thus $\mathcal{E}(R) = \{M \in \mathcal{S} : \delta_M \in R\}$ (cf. (3)). Also $\mathcal{E}_\kappa(R, \theta) = \mathcal{E}(R, \theta) \cap \mathcal{S}_\kappa$.

Example 3.27. Let $R \subseteq \omega_1$ and suppose \mathcal{F}, \mathcal{H} are families of sets of ordinals. Suppose that $\psi(v_0, \dots, v_4)$ is a first order formula such that for every $M \in \mathcal{E}(R)$ containing \mathcal{F} and \mathcal{H} , and every $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$, $\varphi_{\mathcal{T}}(\mathcal{F}, \mathcal{H}, p, a_3)$ implies there

⁴ ZFC minus the Power Set axiom.

is a $y \subseteq M$ such that $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$ holds, and thus equation (10) is satisfied. Then saying Complete has a winning strategy $\mathcal{E}(R)$ - ψ -globally in the game $\mathfrak{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$, means that it has a winning strategy in the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ for all $M \in \mathcal{S}$ with $\mathcal{F}, \mathcal{H}, \mathcal{R}(\mathcal{F}, \mathcal{H}) \in M$ and $\delta_M \in R$, and all $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$. Alternatively, we could have omitted a_3 and equivalently referred to “ $\mathcal{E}_\kappa(R)$ - ψ -globally” instead.

Corollary 3.22.4. *Restricting \mathcal{T} in definition 3.26 to only include families \mathcal{F} of countable sets of ordinals, Extender has nonlosing strategies in the games $\mathfrak{D}_{\text{gen}}$ and $\mathfrak{D}_{\text{gen}}^*$, ψ_{\min} -globally.*

Proof. First we have to show that equation (10) holds. But if $\mathcal{F}, \mathcal{H}, p \in M$ and $\varphi_T(\mathcal{F}, \mathcal{H}, p, a_3)$, then in particular, \mathcal{F} is a family of countable subsets of some ordinal θ and $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})$. Since members of \mathcal{F} are countable, $x \subseteq M$ for all $x \in \mathcal{F} \cap M$ and thus $\psi_{\min}(M, \theta \cap M, \mathcal{F}, \mathcal{H}, p)$ holds.

Now we obtain a nonlosing strategy for Extender in both of the games $\mathfrak{D}_{\text{gen}}(M, \theta \cap M, \mathcal{F}, \mathcal{H}, p)$ and $\mathfrak{D}_{\text{gen}}^*(M, \theta \cap M, \mathcal{F}, \mathcal{H}, p)$ by corollary 3.22.3, since (18) holds. \square

3.3. Complete properness.

Corollary 3.22.5. *Let \mathcal{F} and \mathcal{H} be subfamilies of $[\theta]^{\leq \aleph_0}$, with \mathcal{F} closed under finite reductions. Suppose that $\psi \rightarrow \psi_{\min}$, and that ψ -globally, Extender has no winning strategy in the parameterized game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$. Then $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is completely proper.*

Proof. Suppose $M \prec H_\kappa$ is a countable elementary submodel with $(\mathcal{F}, \mathcal{H}) \in M$. Take $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$. The game $\mathfrak{D}_{\text{gen}}(M, \mathcal{F}, \mathcal{H}, y, p)$ is played with Extender playing p_k on move k according to a nonlosing strategy, which it has by corollary 3.22.4 and proposition 2.14 since $\psi \rightarrow \psi_{\min}$. By the hypothesis that Extender’s strategy in the game $\mathfrak{D}_{\text{gen}}(M, \mathcal{F}, \mathcal{H}, y, p)$ is not a winning strategy, Complete can play in such a way that the game does not result in a win for Extender. Thus the result is a drawn game, and hence $\langle p_k : k < \omega \rangle \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ by proposition 3.12, as required. \square

The following weaker result gives a purely combinatorial characterization of complete properness, unlike corollary 3.22.5.

Corollary 3.22.6. *Let \mathcal{F} and \mathcal{H} be subfamilies of $[\theta]^{\leq \aleph_0}$, with \mathcal{F} closed under finite reductions. Suppose that $\psi \rightarrow \psi_{\min}$, and that ψ -globally, Complete has a winning strategy in the parameterized game $\mathfrak{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$. Then $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is completely proper.*

Proof. By propositions 3.16 and 3.17, Complete has a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, ψ -globally, and in particular, corollary 3.22.5 applies. \square

3.4. Completeness systems. Our formulation of *completeness systems* differs slightly from that observed in the literature. Completeness systems were invented by Shelah, and we use the same underlying ideas as in the original formulation in [She82].

A full account of the theory of completeness systems is given by Abraham in [Abr06]. It is emphasized there that in order to apply the theory, a P_α -name \dot{Q}_α for a poset must be complete for some completeness system that lies in the ground model. Then *simple*—meaning simply definable—completeness systems are introduced to achieve this. An alternative to completeness systems that has gained some popularity was introduced in [ER99], where the necessary combinatorial properties of \dot{Q}_α entailed by the completeness system are isolated. In this approach one directly verifies that the name \dot{Q}_α itself satisfies the prerequisite properties.

Our approach is less robust than in [Abr06]; however, we do not know of any examples not encompassed by our treatment,⁵ and it may have some advantages, including we hope, conceptual simplicity. In our formulation, the fundamental notion is a second order formula rather than the system of filters it describes. Moreover, the completeness system (i.e. this formula) is good for exactly one class of posets. This captures every usage of the completeness systems that we have observed, although there may very well be uses for undefinable (i.e. non-simple) completeness systems, or systems that work for more than one class of posets. A potential advantage of our approach is that, when the formula provably (in ZFC) has the required properties, the completeness system functions in arbitrary forcing extensions, whereas (as indicated in [Abr06]) the approach of using a ground model system is only valid in forcing extensions that do not add new countable subsets of the ground model.

Definition 3.28. We say that a pair of formulae $\wp(v_0, \dots, v_{n-1})$ and $\tau(v_0, \dots, v_{n-1})$ in the language of set theory *describe* a poset over some model N if $N \models \ulcorner \forall x_0, \dots, x_{n-1} \tau(\vec{x}) \rightarrow \wp(\vec{x}) \text{ is a poset} \urcorner$.⁶ If we are working with some ground model V and we say that (\wp, τ) describes a poset, we mean that it describes it over V . And the pair *provably describes* a poset if $\text{ZFC} \vdash \ulcorner \forall \vec{x} \tau(\vec{x}) \rightarrow \wp(\vec{x}) \text{ is a poset} \urcorner$.

Example 3.29. Let $\tau(v_0, v_1)$ express $\ulcorner v_0 \text{ and } v_1 \text{ consist of sets of ordinals} \urcorner$. Then the pair (\mathcal{R}, τ) provably describes a poset, where \mathcal{R} is from definition 3.1, i.e. for $(\mathcal{F}, \mathcal{H})$ satisfying $\tau(\mathcal{F}, \mathcal{H})$, the described poset is $\mathcal{R}(\mathcal{F}, \mathcal{H})$.

Definition 3.30. Suppose (\wp, τ) is a pair of formulae with n free variables that (provably) describes a poset. A (*provable*) *completeness system* for (\wp, τ) will refer to a second order formula $\varphi(Y_0, Y_1; v_0, \dots, v_n)$ for which (it is provable in ZFC that): for every countable $M \prec H_\kappa$, where κ is a sufficiently large regular cardinal, for all $\vec{a} \in M$ satisfying $\tau(\vec{a})$, for every $p \in \wp(\vec{a})^M$ (i.e. $M \models p \in \wp(\vec{a})$), the family of sets

$$\mathcal{G}_Z = \{G \subseteq \text{Gen}(M, \wp(\vec{a}), p) : M \models \varphi(G, Z; \vec{a}, p)\} \quad (Z \subseteq M) \quad (22)$$

- (i) generates a proper filter on $\text{Gen}(M, \wp(\vec{a}), p)$, i.e. every finite intersection $\mathcal{G}_{Z_0} \cap \dots \cap \mathcal{G}_{Z_{n-1}} \subseteq \text{Gen}(M, \wp(\vec{a}), p)$ is nonempty,

⁵ We only deal with σ -complete systems, but our treatment could be adapted to more restrictive systems (e.g. [She98, Ch. VIII, §4]).

⁶ τ is unnecessary but is used for presentational purposes.

- (ii) has a member that is a subset of $\text{Gen}^+(M, \wp(\vec{a}), p)$, i.e. there exists $Z \subseteq M$ such that every element $G \in \mathcal{G}_Z$ has a common extension in $\wp(\vec{a})$.

The completeness system is called σ -complete if (it is provable in ZFC that) for all M , \vec{a} and p as above, the filter generated by the family from equation (22) is σ -complete.

Remark 3.31. To avoid confusion, it should be noted that in the typical formulation from the literature condition (ii) is stipulated by stating that the poset is complete for the given completeness system.

We use α -properness together with completeness systems in what is now a standard method of forcing without adding reals. Since we have made some adjustments to the usual terminology for completeness systems, the following theorem needs to be taken in the present context.

Terminology 3. Let $\vec{P} = (P_\xi, \dot{Q}_\xi : \xi < \mu)$ be an iterated forcing construct.

When we say that an *iterand* \dot{Q}_ξ of \vec{P} satisfies some property Φ we of course mean that $P_\xi \Vdash \Phi(\dot{Q}_\xi)$.

As usual, when we say that \vec{P} has *countable supports* we mean that $P_\delta = \varprojlim_{\xi < \delta} P_\xi$ for limit δ of countable cofinality, and $P_\delta = \varinjlim_{\xi < \delta} P_\xi$ for limit δ of uncountable cofinality. This also determines P_μ for \vec{P} when μ is a limit, and of course $P_\mu = P_{\mu-1} \star \dot{Q}_{\mu-1}$ in case μ is a successor.

Theorem (Shelah). *Let \vec{P} be an iterated forcing construction with countable supports. Suppose that $\mathcal{E} \subseteq [H_\kappa]^{\aleph_0}$ is stationary for some sufficiently large regular cardinal κ ; and suppose for each ξ , (\wp_ξ, τ_ξ) provably describes a poset and has a σ -complete completeness system φ_ξ . If for each $\xi < \mu$, the iterand \dot{Q}_ξ is \mathcal{E} - α -proper, and $\dot{Q}_\xi = \wp_\xi(\vec{a})$ and $\tau_\xi(\vec{a})$ hold for some parameter \vec{a} , then $P_{\text{len}(\vec{P})}$ does not add new reals.*

Definition 3.32. We let \mathbb{D} -complete denote the class of all posets Q for which there exists (\wp, τ) provably describing a poset, such that (\wp, τ) has a σ -complete completeness system and $Q = \wp(\vec{a})$ for some parameter \vec{a} satisfying $\tau(\vec{a})$. Then the forcing axiom $\text{MA}(\mathbb{D}\text{-complete})$ is the statement that for every \mathbb{D} -complete poset Q and every family \mathcal{D} of cardinality $|\mathcal{D}| = \aleph_1$ consisting of dense subsets of Q , there exists a filter $G \subseteq Q$ intersecting every member of \mathcal{D} . We define $\text{MA}(\Phi \text{ and } \mathbb{D}\text{-complete})$ analogously, where Φ is some property of posets (e.g. properness).

Proposition 3.33. *Every poset in \mathbb{D} -complete is completely proper.*

Corollary (Shelah). *$\text{MA}(\alpha\text{-proper and } \mathbb{D}\text{-complete})$ is consistent with CH relative to the consistency of a supercompact cardinal.*

Remark 3.34. Although we have made the effort to distinguish when a formula ψ *provably* has some property, it will not actually have a direct bearing on the topics in this paper. Provability is only needed for the property of describing a poset.

In particular, although, as already mentioned, a completeness system should be in the ground model, simple completeness systems were designed with the following property in mind (adapted to the present context).

Proposition 3.35. *Suppose (\wp, τ) provably describes a poset. The statement $\lceil \wp \text{ is a completeness system for } (\wp, \tau) \rceil$ is absolute between transitive models (of enough of ZFC) that have the same reals. The σ -completeness property is similarly absolute.*

Proof. The point is that the larger model has no new isomorphism types of countable elementary submodels. See e.g. [Abr06]. \square

Definition 3.36. In the context of parameterized games, we may refer to a notion of suitability F as *describing* some type of family of subsets of M . Of course we may do the same when F is given by a formula ψ ; moreover, in the latter case we can say that ψ *provably describes* some family to indicate that this fact is provable in ZFC.

Example 3.37. We say that ψ describes a P -filter, if for all $M \in \mathcal{S}$, for all $\vec{a} \in \mathcal{T} \cap M$, $\{x \subseteq M : \psi(M, x, \vec{a})\}$ is a P -filter on M .

Proposition 3.38. ψ_{\min} provably describes a P -filter (cf. definition 3.26).

Proof. For $M \in \mathcal{S}$ and $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$ the described family is $\{y \subseteq M : x_p \subseteq y, \text{ and } x \subseteq^* y \text{ for all } x \in \mathcal{F} \cap M\}$, which clearly forms a P -filter. \square

Lemma 3.39. *Let ψ be a notion of suitability such that (provably) $\psi \rightarrow \psi_{\min}$. Then there is a (provable) σ -complete completeness system for $\mathcal{R}(\mathcal{F}, \mathcal{H})$ for all subfamilies \mathcal{F} and \mathcal{H} of $[\theta]^{\leq \aleph_0}$ for some ordinal θ with \mathcal{F} closed under finite reductions for which ψ -globally, Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$.*

Remark 3.40. Lemma 3.39 is asserting the existence of a completeness system for (\mathcal{R}, τ) where $\tau(\mathcal{F}, \mathcal{H})$ expresses $\lceil \mathcal{F}$ and \mathcal{H} are families of countable sets of ordinals with \mathcal{F} closed under finite reductions such that ψ -globally, Complete has a forward nonlosing strategy for $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H}) \rceil$.

Proof. We fix a definable method of coding

- a subset y of $\theta \cap M$,
- for each $t \in \text{Fin}(\theta) \cap M$, a function $\Phi(t)$ with domain a subset of M and codomain $\text{Fin}(\theta) \cap M$,

by subsets $Z \subseteq M$. Then we let $\varphi(G, Z; \mathcal{F}, \mathcal{H}, p)$ be a formula expressing \lceil if both

- (a) $\phi_{\min}(y; \mathcal{F}, \mathcal{H}, p)$ (cf. definition 3.24),
- (b) $\Phi(t)$ is a forward strategy for Complete in the game $\mathfrak{D}_{\text{gen}}(M, y \cup t, \mathcal{F}, \mathcal{H}, p)$ for all $t \in \text{Fin}(\theta)$ such that $\phi_{\min}(y \cup t; \mathcal{F}, \mathcal{H}, p)$,

then G is the filter generated by $(p_k : k < \omega)$, where $(p_k : k < \omega)$ is some sequence satisfying: there exists $m < \omega$ and $t \in \text{Fin}(\theta)$ such that

- (c) $\phi_{\min}(y \cup t; \mathcal{F}, \mathcal{H}, p)$,

- (d) the game $\mathcal{D}_{\text{gen}}(M, y \cup t, \mathcal{F}, \mathcal{H}, p)$ is played and (p_k, s_k) denotes move k ,
- (e) Complete plays $s_k \supseteq \Phi(t)(P_k)$ for all $k \geq m$, where P_k is the position after Extender's k^{th} move⁷,

where y and Φ are the objects coded by Z .⁷

Assuming that $\vec{a} = (\mathcal{F}, \mathcal{H})$ satisfies the hypotheses, then given $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})^M$ we need to check that the family of subsets of M given by

$$\mathcal{G}_Z = \{G \in \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p) : M \models \varphi(G, Z; \mathcal{F}, \mathcal{H}, p)\} \quad (Z \subseteq M) \quad (23)$$

has the required properties. First we note that \mathcal{G}_Z is a nonempty subset of $\text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ whenever $M \models \varphi(G, Z; \mathcal{F}, \mathcal{H}, p)$. This is because when clauses (a) and (b) are true, letting y and Φ be the objects coded by Z , $y \supseteq x_p$ bounds $\mathcal{F} \cap M$ by proposition 3.25, and $\Phi(\emptyset)$ is a (forward) strategy for Complete in the game $\mathcal{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$. Thus setting $m = 0$ and $t = \emptyset$, Extender can play according to a nonlosing strategy for $\mathcal{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$ by corollary 3.22.4 and Complete can play valid moves $\Phi(\emptyset)(P_k)$, proving that a sequence $(p_k : k < \omega)$ satisfying (c)–(e) does exist, and moreover G is generic over M because Extender does not lose.

Next we show that the family forms a σ -complete filter base, which in particular establishes (i). We in fact establish the stronger property that its upwards closure is a σ -complete filter. This is of course done by diagonalizing the coded objects. Take $Z_n \subseteq M$ ($n < \omega$) and assume without loss of generality that (a) and (b) are satisfied for all n . For each n , let y_n and Φ_n be the objects coded by Z_n . Since ψ_{\min} describes σ - \supseteq^* -directed family, using proposition 3.25 there exists $y_\omega \subseteq M$ such that $M \models \phi_{\min}(y_\omega; \mathcal{F}, \mathcal{H}, p)$ and $y_\omega \subseteq^* y_n$ for all $n < \omega$. Fix some enumeration $(u_j : j < \omega)$ of $\text{Fin}(\theta) \cap M$. For each $t \in \text{Fin}(\theta) \cap M$ such that $M \models \phi_{\min}(y_\omega \cup t; \mathcal{F}, \mathcal{H}, p)$, let

$$\Psi(t)(P) = \bigcup \{ \Phi_i(u_j)(P) : i, j < |P|, M \models \phi_{\min}(y_i \cup u_j; \mathcal{F}, \mathcal{H}, p), \\ P \text{ is a position of the game } \mathcal{D}_{\text{gen}}(M, y_i \cup u_j, \mathcal{F}, \mathcal{H}, p) \} \quad (24)$$

for every position P in the game $\mathcal{D}_{\text{gen}}(M, y_\omega \cup t, \mathcal{F}, \mathcal{H}, p)$ for which it is Complete's turn to play (its $|P| - 1^{\text{th}}$ move), where an empty union is taken to be the empty set, and then set

$$\Phi_\omega(t)(P) = \Psi(t)(P) \cap (y_\omega \cup t). \quad (25)$$

Then letting $Z_\omega \subseteq M$ code y_ω and Φ_ω , clearly (a) and (b) hold for Z_ω . We will show that $\mathcal{G}_{Z_\omega} \subseteq \bigcap_{n=0}^{\infty} \mathcal{G}_{Z_n}$. To see this, suppose $(p_k : k < \omega)$ is a sequence satisfying (c)–(e) for Z_ω , witnessed by $m < \omega$, $t \in \text{Fin}(\theta) \cap M$ and Complete's moves $(s_k : k < \omega)$. Given n , we need to show that $(p_k : k < \omega)$ satisfies (c)–(e) for Z_n . Since ψ_{\min} describes a filter, $M \models \phi_{\min}(y_n \cup y_\omega \cup t; \mathcal{F}, \mathcal{H}, p)$. Then letting $t' \in \text{Fin}(\theta) \cap M$ satisfy

$$y_n \cup t' = y_n \cup y_\omega \cup t, \quad (26)$$

⁷ More precisely, we fix a method of coding sequences of elements of M by subsets of M , and then φ is of the form $\ulcorner \exists Y \rho(G, Z, Y; \mathcal{F}, \mathcal{H}, p) \urcorner$ where ρ has no second order quantifiers and Y is used to code the game played, i.e. it codes $(p_k, s_k : k < \omega)$.

t' witnesses that (c) holds. Put $m' = \max\{m, n, j\}$ where $u_j = t'$. Then satisfaction of the conditions for Z_n is witnessed by m' and t' , since (26) implies that (d) holds and for each $k \geq m$, $\Psi(t)(P_k) \cap (y_n \cup t')$ is a valid move in the game $\mathcal{D}_{\text{gen}}(M, y_n \cup t', \mathcal{F}, \mathcal{H}, p)$, where P_k is the position in the game $\mathcal{D}_{\text{gen}}(M, y_\omega \cup t, \mathcal{F}, \mathcal{H}, p)$ after Extender's k^{th} move (and thus $|P_k| = k + 1$), because by (25), $\Psi(t)(P_k) \cap (y_\omega \cup t) \subseteq s_k$. Now $\Psi(t)(P_k) \supseteq \Phi_n(t')(P_k)$ for all $k \geq \max\{n, j\}$ proving (e) for Z_n with $s_k := \Psi(t)(P_k) \cap (y_n \cup t')$ for $k = m, m + 1, \dots$.

It remains to verify (ii). Indeed the requirement of equation (10) guarantees a $y \subseteq M$ satisfying $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$; and by assumption, Complete has a forward nonlosing strategy $\Phi(t)$ in the game $\mathcal{D}_{\text{gen}}(M, y \cup t, \mathcal{F}, \mathcal{H}, p)$ for all $t \in \text{Fin}(\theta) \cap M$ satisfying $\psi(M, y \cup t, \mathcal{F}, \mathcal{H}, p)$. Let $Z \subseteq M$ code y and Φ . Then (a) and (b) hold since $\psi \rightarrow \psi_{\min}$, and thus every member $G \in \mathcal{G}_Z$ is generated by $\langle p_k : k < \omega \rangle$ resulting from the game $\mathcal{D}_{\text{gen}}(M, y \cup t, \mathcal{F}, \mathcal{H}, p)$ being played for some t . Since Extender does not lose as $\langle p_k : k < \omega \rangle = G \in \text{Gen}(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$, and since Complete plays supersets of the forward nonlosing strategy $\Phi(t)$ for all but finitely many moves by (e), the game results in a draw by proposition 3.19. Therefore $G = \langle p_k : k < \omega \rangle \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ by proposition 3.12. \square

3.5. Upwards boundedly order closed subfamilies. In [Hir07a] we introduced the following weakening of order closedness.

Definition 3.41. We say that a subset A of a poset P is *upwards boundedly order closed* if for every nonempty $B \subseteq A$ with an upper bound $p \in P$ (i.e. $b \leq p$ for all $b \in B$): if B has a supremum $\bigvee B$ in P , then $\bigvee B \in A$.

In the present context of subfamilies $\mathcal{H} \subseteq \mathcal{P}(\theta)$, we say that \mathcal{H} is *upwards boundedly order closed* to indicate that it is so in the tree $(\mathcal{P}(\theta), \sqsubseteq)$ of initial segments.

There is a simple criterion for it.

Lemma 3.42. *Every convex (cf. §2.1) subset of a poset is upwards boundedly order closed.*

Proof. Let (P, \leq) be a poset, and let $C \subseteq P$ be convex. Take a nonempty $B \subseteq C$, say $b \in B$, with an upper bound $p \in P$. Suppose B has a supremum $\bigvee B$ in P . Then $b \leq \bigvee B \leq p$ implies $\bigvee B \in C$ by convexity. \square

Corollary 3.42.1. *If \mathcal{H} is a convex subfamily of $(\mathcal{P}(\theta), \sqsubseteq)$ then \mathcal{H} is upwards boundedly order closed.*

Applying the definition in the present context gives:

Proposition 3.43. *\mathcal{H} is upwards boundedly order closed iff every nonempty subfamily $\mathcal{K} \subseteq \mathcal{H}$ with a $y \in \mathcal{H}$, such that $x \sqsubseteq y$ for all $x \in \mathcal{K}$, satisfies $\bigcup \mathcal{K} \in \mathcal{H}$.*

This endows the poset $\mathcal{R}(\mathcal{F}, \mathcal{H})$ with the following crucial property.

Proposition 3.44. *Let \mathcal{F}, \mathcal{H} be families of subsets of θ , with \mathcal{H} upwards boundedly order closed. If a family $Q \subseteq \mathcal{R}(\mathcal{F}, \mathcal{H})$ has a common extension in $\mathcal{R}(\mathcal{F}, \mathcal{H})$, then it has a common extension q such that*

$$x_q = \bigcup_{p \in Q} x_p. \quad (27)$$

In the present context we consider a generalization.

Definition 3.45. Let $\mathcal{H} \subseteq \mathcal{P}(\theta)$ and $R \subseteq \theta$. We say that \mathcal{H} is *upwards boundedly order closed beyond R* if $\bigcup \mathcal{K} \in \mathcal{H}$ whenever $\mathcal{K} \subseteq \mathcal{H}$ is a nonempty subfamily with $y \in \mathcal{H}$ such that $x \sqsubseteq y$ for all $x \in \mathcal{K}$ and such that

$$\sup\left(\bigcup \mathcal{K}\right) \notin R. \quad (28)$$

Proposition 3.46. *Let \mathcal{F}, \mathcal{H} be families of subsets of θ , with \mathcal{H} upwards boundedly order closed beyond $R \subseteq \theta$. If a family $Q \subseteq \mathcal{R}(\mathcal{F}, \mathcal{H})$ has a common extension in $\mathcal{R}(\mathcal{F}, \mathcal{H})$ and*

$$\sup\left(\bigcup_{p \in Q} x_p\right) \notin R, \quad (29)$$

then it has a common extension q such that $x_q = \bigcup_{p \in Q} x_p$.

This property of the family \mathcal{H} allows us to obtain α -properness for the poset $\mathcal{R}(\mathcal{F}, \mathcal{H})$. It will also be used in the next section (§3.6) to obtain a strong chain condition for the poset.

Claim 1. *Let \mathcal{F} and \mathcal{H} be subfamilies of $[\theta]^{\leq \aleph_0}$, with \mathcal{F} closed under finite reductions and \mathcal{H} upwards boundedly order closed. Suppose $\mathcal{F}, \mathcal{H} \in M \prec H_\kappa$ is countable, $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$ and $y \subseteq M$ is a \subseteq^* -bound of $\mathcal{F} \cap M$ with $x_p \subseteq y$. If Extender does not have a winning strategy in the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$, then there exists $q \in \text{gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ such that $x_q \subseteq y$.*

Proof. The game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$ is played with Extender playing according to a nonlosing strategy by corollary 3.22.4. Since Extender's strategy is not a winning strategy, Complete can play so that Extender does not win, and hence the game is drawn. Then Extender's sequence of moves $(p_k : k < \omega)$ generates a completely $(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ -generic filter G , say with extension $q \in \mathcal{R}(\mathcal{F}, \mathcal{H})$, by proposition 3.12. And by proposition 3.44, we may assume that $x_q = \bigcup_{\bar{p} \in G} x_{\bar{p}} = \bigcup_{k < \omega} x_{p_k} \subseteq y$ as needed. \square

Claim 2. *Let \mathcal{F} and \mathcal{H} be subfamilies of $[\theta]^{\leq \aleph_0}$, with \mathcal{F} closed under finite reductions and \mathcal{H} upwards boundedly order closed beyond $R \subseteq \theta$, such that \mathcal{H} is \mathcal{F} -extendable. Suppose $\mathcal{F}, \mathcal{H} \in M \prec H_\kappa$ is countable with $\sup(\theta \cap M) \notin R$, $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$ and $y \subseteq M$ is a \subseteq^* -bound of $\mathcal{F} \cap M$ with $x_p \subseteq y$. If Extender has no winning strategy in the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$, then there exists $q \in \text{gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ such that $x_q \subseteq y$.*

Proof. Set $\delta = \sup(\theta \cap M)$. We proceed as in the proof of claim 1, but now since \mathcal{H} is \mathcal{F} -extendable, we also have by proposition 3.6 that for every $\xi < \theta$

in M , $\mathcal{D}_\xi \in M$ (cf. equation (14)) is dense, and thus $x_{p_k} \in \mathcal{D}_\xi$ for some k since Extender did not lose. This entails that $\sup(\bigcup_{k < \omega} x_{p_k}) = \delta$, and since $\delta \notin R$ by assumption, we use proposition 3.46 to obtain the desired extension q with $x_q \subseteq y$. \square

Definition 3.47. We say that a suitability function F is *coherent* if for all $M \in N$ in \mathcal{S} , and all $\vec{a} \in \mathcal{T} \cap M$, every $y \in F(N)(\vec{a})$ has an $x \subseteq y \cap M$ in $F(M)(\vec{a}) \cap N$. When the function is given by a formula ψ we say that ψ is *provably coherent*, as usual to specify that coherence is provable in ZFC.

Proposition 3.48. *If we restrict \mathcal{T} in definition 3.26 so that $(\mathcal{F}, \mathcal{H}, p) \in \mathcal{T}$ only if $(\mathcal{F}, \subseteq^*)$ is a σ -directed family of countable sets of ordinals, then ψ_{\min} is provably coherent.*

Proof. Suppose $M \in N$, $\mathcal{F}, \mathcal{H}, p \in M$ and $\psi_{\min}(N, y, \mathcal{F}, \mathcal{H}, p)$. Since \mathcal{F} is σ -directed and M is countable, by proposition 3.3, there exists $x_p \subseteq y' \in \mathcal{F}$ bounding $\mathcal{F} \cap M$, and by elementarity we can find such a $y' \in N$. Now $y' \subseteq^* y$, and $y' \subseteq N$ as y' is countable, and thus $y' \cap y \in N$. Hence $y' \cap y \cap M \subseteq y$ is in N and clearly $\psi_{\min}(M, y' \cap y \cap M, \mathcal{F}, \mathcal{H}, p)$ holds. \square

Definition 3.49. Let ψ be a formula that is to be used in definition 3.26. We say that ψ *respects* \mathcal{D}_{gen} if for all $M \in \mathcal{S}$, all $\mathcal{F}, \mathcal{H} \in M$ and all $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$, if $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$ and $(p_0, s_0), \dots, (p_k, s_k)$ is a position in the game $\mathcal{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$, then $\psi(M, y \setminus \bigcup_{i=0}^k s_i, \mathcal{F}, \mathcal{H}, p_k)$. We may also specify that ψ *provably respects* \mathcal{D}_{gen} .

Proposition 3.50. *For any pair of families $(\mathcal{F}, \mathcal{H})$, ψ_{\min} provably respects \mathcal{D}_{gen} .*

Proof. Immediate from the definitions. \square

3.5.1. α -properness.

Lemma 3.51. *Let \mathcal{F} be a subfamily of $[\theta]^{\leq \aleph_0}$ closed under finite reductions, and let \mathcal{H} be an upwards boundedly order closed subset of $([\theta]^{\leq \aleph_0}, \sqsubseteq)$. Suppose that $\psi \rightarrow \psi_{\min}$ is coherent and respects \mathcal{D}_{gen} . If Complete has a nonlosing strategy for $\mathcal{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, \mathcal{E} - ψ -globally, then $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is \mathcal{E} - α -proper.*

Proof. The proof is by induction on $\alpha < \omega_1$. The induction hypothesis is:

(IH $_\beta$) For every tower $M_0 \in M_1 \in \dots$ of members of \mathcal{E} , that are also elementary submodels of H_κ , of height $\beta + 1$ with $\mathcal{F}, \mathcal{H} \in M_0$, every $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M_0$ and every $y \subseteq M_\beta$ such that $\psi(M_\beta, y, \mathcal{F}, \mathcal{H}, p)$, there exists $q \in \text{gen}^+(\{M_\xi : \xi \leq \beta\}, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ with $x_q \subseteq y$.

Assume that (IH $_\beta$) holds for all $\beta < \alpha$.

Suppose $M_0 \in M_1 \in \dots$ is a tower in \mathcal{E} of elementary submodels of H_κ of height $\alpha + 1$, with $\mathcal{F}, \mathcal{H} \in M_0$, and take $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M_0$. Suppose we are given $y \subseteq M_\alpha$ satisfying $\psi(M_\alpha, y, \mathcal{F}, \mathcal{H}, p)$. In the case $\alpha = 0$, (IH $_0$) follows immediately from claim 1 because $\psi \rightarrow \psi_{\min}$ and Complete has a nonlosing strategy in the game $\mathcal{D}_{\text{gen}}(M_\alpha, y, \mathcal{F}, \mathcal{H}, p)$.

Consider the case $\alpha = \beta + 1$ is a successor. Since ψ is coherent, there exists $y' \subseteq y \cap M_\beta$ in M_α satisfying $\psi(M_\beta, y', \mathcal{F}, \mathcal{H}, p)$. Now applying (IH $_\beta$) in the

model M_α with $y := y'$, we obtain $q' \in \text{gen}^+(\{M_\xi : \xi \leq \beta\}, \mathcal{R}(\mathcal{F}, \mathcal{H}), p) \cap M_\alpha$ with $x_{q'} \subseteq y$. Then we use claim 1 with $M := M_\alpha$ and $p := q'$ to extend q' to q completely generic over M_α with $x_q \subseteq y$.

Assume now that α is a limit, say $(\eta_k : k < \omega)$ is a strictly increasing sequence cofinal in α . The game $\mathfrak{D}_{\text{gen}}(M_\alpha, y, \mathcal{F}, \mathcal{H}, p)$ is played with Complete using its nonlosing strategy. After $k + 1$ moves $(p_0, s_0), \dots, (p_k, s_k)$ have been played, assume that $p_k \in M_{\eta_{k-1}+1}$ (taking $\eta_{-1} + 1 = 0$) is completely generic over $\{M_\xi : \xi \leq \eta_{k-1}\}$ with $x_{p_k} \subseteq y$. Since ψ respects $\mathfrak{D}_{\text{gen}}$ and $(p_0, s_0), \dots, (p_k, s_k)$ is a valid position in the game $\mathfrak{D}_{\text{gen}}(M_{\eta_k}, y, \mathcal{F}, \mathcal{H}, p)$, $\psi(M_{\eta_k}, y \setminus \bigcup_{i=0}^k s_i, \mathcal{F}, \mathcal{H}, p_k)$ holds. And by coherence, there exists $y' \subseteq y \setminus \bigcup_{i=0}^k s_i \cap M_{\eta_k}$ in $M_{\eta_{k+1}}$ satisfying $\psi(M_{\eta_k}, y', \mathcal{F}, \mathcal{H}, p_k)$. Extender can now make a move $p_{k+1} \in M_{\eta_{k+1}}$ completely generic over $\{M_\xi : \xi \leq \eta_k\}$ with $x_{p_{k+1}} \subseteq y$ by applying $(\text{IH}_{\eta_k - (\eta_{k-1} + 1)})$ in the model $M_{\eta_{k+1}}$ to the tower $M_{\eta_{k-1}+1} \in \dots \in M_{\eta_k}$, $p := p_k$ and $y := y' \setminus \bigcup_{i=0}^k s_k$.

The continuity of the \in -chain ensures that Extender does not lose the game. And since Complete does not lose, $p_0 \leq p_1 \leq \dots$ has a common extension q with $x_q \subseteq y$ by proposition 3.44. Since q is completely generic over $\{M_\xi : \xi \leq \alpha\}$ the induction is complete.

To see that $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is \mathcal{E} - α -proper, suppose $M_0 \in \dots \in M_\beta$ is a tower in \mathcal{E} with $\mathcal{F}, \mathcal{H} \in M_0$, and take $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M_0$. Equation (8) gives a $y \subseteq M_\beta$ such that $\psi(M_\beta, y, \mathcal{F}, \mathcal{H}, p)$. Then (IH_β) implies the existence of $q \geq p$ generic over every member of the tower. \square

Lemma 3.52. *Let \mathcal{F} be a subfamily of $[\theta]^{\leq \aleph_0}$ closed under finite reductions, and let $\mathcal{H} \subseteq [\theta]^{\leq \aleph_0}$ be an upwards boundedly order closed beyond $R \subseteq \theta$ that is \mathcal{F} -extendable. Suppose that $\psi \rightarrow \psi_{\min}$ is coherent and respects $\mathfrak{D}_{\text{gen}}$. If $\mathcal{E} \subseteq \mathcal{E}(\theta \setminus R, \theta)$ and Complete has a nonlosing strategy for $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, \mathcal{E} - ψ -globally, then $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is \mathcal{E} - α -proper.*

Proof. The proof is the same as the proof of lemma 3.51, except that we use claim 2 in place of claim 1. This is justified by the definition of \mathcal{E} , because \mathcal{H} is \mathcal{F} -extendable. \square

We shall require the following basic observation on an equivalent of \mathcal{E} - α -properness. Cf. equation (21) for the notation.

Lemma 3.53. *Let $S \subseteq \theta$ be stationary for some regular cardinal θ . If a forcing notion P is $\mathcal{E}(S \setminus A, \theta)$ - α -proper for some $A \in \text{NS}_\theta$, then P is $\mathcal{E}(S, \theta)$ - α -proper.*

Proof. This is because for any countable $M \prec H_\kappa$ with $P \in M$, we may assume that $A \in M$, and thus $\sup(\theta \cap M) \notin A$. Hence the set A does not interfere with $\mathcal{E}(S)$ - α -properness. \square

3.6. Isomorphic models. We introduce a new variant of MA here in definition 3.58, that is consistent with CH relative to ZFC. It is based on Shelah's theorem below ([She98, Ch. VIII, Lemma 2.4]) for obtaining \aleph_2 -cc iterations. Alternatively, we could have used the appropriate axiom from [She98, Ch. VIII, §3].

This will only be possible for the case $\theta = \omega_1$, and we still need some theorem in this case because in general for $\mathcal{F}, \mathcal{H} \subseteq [\omega_1]^{\leq \aleph_0}$, the cardinality of our poset

is large:

$$|\mathcal{R}(\mathcal{F}, \mathcal{H})| = 2^{\aleph_1^{\aleph_0}} \geq \aleph_2 \quad (30)$$

(and equal to \aleph_2 under GCH).

Definition 3.54. We say that a suitability function F (as in definition 2.9) *respects isomorphisms* if for every two isomorphic models M and N in \mathcal{S} , for every (first order) isomorphism $h : M \rightarrow N$ fixing $M \cap N$,

$$x \in F(M)(\vec{a}) \quad \text{iff} \quad h[x] \in F(N)(h(\vec{a}))$$

for all $x \subseteq M$ and all $\vec{a} \in \mathcal{T} \cap M$. (31)

If ψ is a formula describing a suitability function, then we say that ψ (provably) *respects isomorphisms* if (it is provable that) the function described by ψ respects isomorphisms.

Example 3.55. In the context of definition 3.26, with \mathcal{F} and \mathcal{H} fixed, ψ respects isomorphisms iff for all $M, N \in \mathcal{S}$, and all isomorphisms $h : M \rightarrow N$ fixing $M \cap N$,

$$\psi(M, y, \mathcal{F}, \mathcal{H}, p) \quad \text{iff} \quad h[y] \in \psi(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p))$$

for all $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$. (32)

Proposition 3.56. *Provided that we restrict \mathcal{T} to only include families \mathcal{F} of countable sets of ordinals, ψ_{\min} provably respects isomorphisms.*

Proof. Let $h : M \rightarrow N$ be an isomorphism. Assume $\psi_{\min}(M, y, \mathcal{F}, \mathcal{H}, p)$, i.e. $x_p \subseteq y$ and $y \subseteq M$ is a \subseteq^* -bound of $\mathcal{F} \cap M$. Then $h[y]$ is a \subseteq^* -bound of $h(\mathcal{F}) \cap N$ and $x_{h(p)} = h(x_p) \subseteq h[y]$, because $x \subseteq M$ for all $x \in \mathcal{F} \cap M$, yielding $\psi_{\min}(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p))$. \square

Lemma 3.57. *Let \mathcal{F} and \mathcal{H} be subfamilies of $[\theta]^{\leq \aleph_0}$ with \mathcal{F} closed under finite reductions. Suppose that $\psi \rightarrow \psi_{\min}$, and ψ -globally, Complete has a nonlosing strategy for $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$. Then for any two isomorphic countable models $M, N \prec H_\kappa$, say $h : M \rightarrow N$ is an isomorphism, with $\mathcal{F}, \mathcal{H} \in M$, if*

$$\psi(M, y, \mathcal{F}, \mathcal{H}, p) \quad \text{iff} \quad \psi(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p))$$

for all $y \subseteq M$ and all $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$, (33)

then for every $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$, there exists $G \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ such that $h[G] \in \text{Gen}^+(N, \mathcal{R}(h(\mathcal{F}), h(\mathcal{H})), h(p))$.

Proof. Suppose we are given two isomorphic countable models $M, N \prec H_\kappa$, say $h : M \rightarrow N$ is an isomorphism, with $\mathcal{F}, \mathcal{H} \in M$, and $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$. By equation (10), there exists $y \subseteq M$ such that $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$. Thus $\psi(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p))$ by equation (33). Also note that ψ -globally, Complete has a nonlosing strategy for $\mathfrak{D}_{\text{gen}}(h(\mathcal{F}), h(\mathcal{H}))$ by elementarity.

The games $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$ and $\mathfrak{D}_{\text{gen}}(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p))$ are played simultaneously, and we let Φ and Φ' denote nonlosing strategies for Complete in the respective games. Extender plays p_k on its k^{th} move in the game

$\mathcal{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$, according to a nonlosing strategy which it has by corollary 3.22.4 as $\psi \rightarrow \psi_{\min}$. And

$$\text{Extender plays } h(p_k) \text{ in the game } \mathcal{D}_{\text{gen}}(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p)); \quad (34)$$

the validity of this move is verified below. On its k^{th} move, Complete plays

$$s_k \cup h^{-1}(t_k) \quad \text{where } s_k = \Phi(P_k) \text{ and } t_k = \Phi'(h(P_k)) \quad (35)$$

in the game $\mathcal{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{H}, p)$, where P_k is the position after Extender's k^{th} move, and plays $h(s_k) \cup t_k$ in the game $\mathcal{D}_{\text{gen}}(N, h[y], h(\mathcal{F}), h(\mathcal{H}), h(p))$. Note that $t_k \subseteq h[y] \setminus h(x_{p_k})$ by (34), which implies $h^{-1}(t_k) \subseteq y \setminus x_{p_k}$, and thus $s_k \cup h^{-1}(t_k)$ is a valid move for Complete in the former game, and similarly $h(s_k) \cup t_k$ is a valid move in the latter game. Also note that Extender's move $h(x_{p_k})$ is valid in the latter game: (1) $h(p_k) \geq h(p_{k-1})$ and (2) $h(x_{p_k}) \subseteq h[y] \setminus \bigcup_{i=0}^{k-1} h(s_i) \cup t_i$ as $x_{p_k} \subseteq y \setminus \bigcup_{i=0}^{k-1} s_i \cup h^{-1}(t_i)$.

After the games, let G be the filter of $\mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$ generated by $(p_k : k < \omega)$, and let H be the filter of $\mathcal{R}(h(\mathcal{F}), h(\mathcal{H})) \cap N$ generated by $(h(p_k) : k < \omega)$, so that $H = h[G]$. By proposition 3.19, Complete does not lose either games. Extender does not lose the former game since it played according to a nonlosing strategy, and it does not lose the latter game, because for every dense $D \subseteq \mathcal{R}(h(\mathcal{F}), h(\mathcal{H}))$ in N , $p_k \in h^{-1}(D)$ for some k , and thus $h(p_k) \in D$. Therefore, both $G \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ and $h[G] \in \text{Gen}^+(N, \mathcal{R}(\mathcal{F}, \mathcal{H}), h(p))$ by proposition 3.12. \square

Recall that a poset P satisfies the *properness isomorphism condition* if for every two isomorphic countable $M, N \prec H_\kappa$, for κ a sufficiently large regular cardinal, via $h : M \rightarrow N$ with $P \in M \cap N$ and h fixing $M \cap N$, for every $p \in P \cap M$, there exists $G \in \text{Gen}^+(M, P, p)$ such that $h[G] \in \text{Gen}^+(N, P, h(p))$ and moreover there exists $q \in P$ extending both G and $h[G]$ (this is the \aleph_2 -pic from [She98, Ch. VIII, §2]).

Theorem (Shelah). *Assume CH. Let $\vec{P} = (P_\xi, \dot{Q}_\xi : \xi < \mu)$ be a countable support iterated forcing construction of length $\mu \leq \omega_2$. If each iterand satisfies the properness isomorphism condition then P_μ satisfies the \aleph_2 -chain condition.*

Definition 3.58. We write $\text{MA}(\alpha\text{-proper} + \mathbb{D}\text{-complete} + \text{pic} + \Delta_0\text{-}H_{\aleph_2}\text{-definable})$ to be interpreted as in definition 3.32, where *pic* denotes the class of posets satisfying the properness isomorphism condition, and $\Delta_0\text{-}H_{\aleph_2}\text{-definable}$ denotes those posets P that are Δ_0 definable over H_{\aleph_2} , i.e. there exists a Δ_0 formula (in the Lévy hierarchy) $\varphi(v_0, \dots, v_n)$ and parameters $a_1, \dots, a_n \in H_{\aleph_2}$ such that $P = \{x \in H_{\aleph_2} : H_{\aleph_2} \models \varphi[x, a_1, \dots, a_n]\}$.

Example 3.59. Our class \mathcal{R} of posets is clearly Δ_0 . Thus $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is $\Delta_0\text{-}H_{\aleph_2}$ -definable whenever \mathcal{F}, \mathcal{H} are families of subsets of $\theta < \omega_2$.

Theorem 3.1. *$\text{MA}(\alpha\text{-proper} + \mathbb{D}\text{-complete} + \text{pic} + \Delta_0\text{-}H_{\aleph_2}\text{-definable})$ is consistent with CH relative to ZFC.*

Proof. Beginning with ground model satisfying GCH, we construct an iteration $(P_\xi, \dot{Q}_\xi : \xi < \omega_2)$ of forcing notions of length ω_2 . At every stage ξ of the iteration it is arranged that

- (i) $P_\xi \Vdash \text{GCH}$,
- (ii) P_ξ has a dense suborder of cardinality at most \aleph_2 ,
- (iii) P_ξ has the \aleph_2 -cc.

This entails that there are \aleph_2 many P_ξ -names for members of H_{\aleph_2} , where H_{\aleph_2} is being interpreted in the forcing extension. Using standard bookkeeping methods, we can arrange a sequence $(\varphi_\xi, \dot{a}_\xi : \xi < \omega_2)$ where φ_ξ is a Δ_0 formula and \dot{a}_ξ is a P_ξ -name for a parameter in H_{\aleph_2} , so that regarding P_ξ -names as also being P_η -names for $\xi \leq \eta$, every pair (φ, \dot{a}) where \dot{a} is a P_ξ -name appears as $(\varphi_\eta, \dot{a}_\eta)$ for some $\eta \geq \xi$. By skipping steps if necessary, we may assume that P_ξ forces that \dot{Q}_ξ is the object defined by $(\varphi_\xi, \dot{a}_\xi)$ over H_{\aleph_2} , and that \dot{Q}_ξ is an α -proper poset in the class \mathbb{D} -complete (cf. definition 3.32), with the pair of formulae describing \dot{Q}_ξ fixed in V ,⁸ and is in pic.

Since every $P_\xi \Vdash |\dot{Q}_\xi| \leq \aleph_2$ and \dot{Q}_ξ is in pic, it follows from Shelah's Theorem above that (i)–(iii) hold at every intermediate stage of the iteration, and moreover that $P_{\omega_2} = \varinjlim_{\xi < \omega_2} P_\xi$ has the \aleph_2 -cc. By Shelah's Theorem on page 24, P_{ω_2} does not add any reals, and thus CH holds in its forcing extension.

In the forcing extension by P_{ω_2} : Suppose that φ is a Δ_0 formula and $a \in H_{\aleph_2}$ so that (φ, a) defines an α -proper poset Q in \mathbb{D} -complete and pic over H_{\aleph_2} , and suppose that $\mathcal{D} \subseteq Q$ is a family of dense subsets of cardinality $|\mathcal{D}| = \aleph_1$. By the \aleph_2 -cc, there exists ξ such that \mathcal{D} is in the intermediate extension and $P_{\omega_2} \Vdash (\varphi_\xi, \dot{a}_\xi) = (\varphi, a)$. In the intermediate extension by P_ξ : The interpretation Q_ξ of the iterand \dot{Q}_ξ is given by $Q_\xi = \{x \in H_{\aleph_2} : H_{\aleph_2} \models \varphi[x, a]\}$. Since Δ_0 formulae are absolute between transitive models, it follows that Q_ξ is a suborder of Q . Thus $D \cap Q_\xi$ is dense for all $D \in \mathcal{D}$. And at the ξ^{th} stage we force a filter intersecting $D \cap Q_\xi \subseteq D$ for all $D \in \mathcal{D}$. Notice that α -properness is downwards absolute, and by proposition 3.35, Q_ξ is also in \mathbb{D} -complete in the intermediate model. \square

Remark 3.60. The same proof shows that we can allow posets whose base set is Σ_1 - H_{\aleph_2} -definable but the ordering still must be Δ_0 - H_{\aleph_2} -definable.

It is sometimes possible to obtain $\mathcal{R}(\mathcal{F}, H)$ with the properness isomorphism condition for $\mathcal{F}, \mathcal{H} \subseteq [\omega_1]^{\leq \aleph_0}$ for the following reason.

Proposition 3.61. *For $M, N \prec H_\kappa$ and $h : M \rightarrow N$ an isomorphism, $h(\alpha) = \alpha$ for all $\alpha < \omega_1$. Thus $[\omega_1]^{\leq \aleph_0} \cap M = h[[\omega_1]^{\leq \aleph_0} \cap M] = [\omega_1]^{\leq \aleph_0} \cap N$.*

Proof. This immediately follows from the fact that $\alpha \subseteq M$ for every countable ordinal $\alpha \in M$. \square

⁸ This is a subtle point. To apply Shelah's Theorem on page 24, we need to know the pair of formulae describing \dot{Q}_ξ in the ground model, i.e. a P_ξ -name for a pair of formulae does not suffice. In practice, however, this does not pose any difficulties.

We will see that for some families $\mathcal{F}, \mathcal{H} \subseteq [\omega_1]^{\leq \aleph_0}$, lemma 3.57 is already enough to establish the properness isomorphism condition.

Corollary 3.56.1. *Let $\mathcal{F}, \mathcal{H} \subseteq [\omega_1]^{\leq \aleph_0}$ be subfamilies with \mathcal{F} closed under finite reductions and \mathcal{H} upwards boundedly order closed. If $\psi \rightarrow \psi_{\min}$ respects isomorphisms for the fixed pair $(\mathcal{F}, \mathcal{H})$ and ψ -globally, Complete has a nonlosing strategy for $\mathcal{O}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, then $\mathcal{R}(\mathcal{F}, \mathcal{H})$ satisfies the properness isomorphism condition.*

Proof. Suppose $M, N \prec H_\kappa$ are countable, $h : M \rightarrow N$ is an isomorphism fixing $M \cap N$ and $\mathcal{F}, \mathcal{H} \in M \cap N$. Since h fixes \mathcal{F} and \mathcal{H} , the fact that ψ respects isomorphisms entails equation (33). Take $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$. Then by lemma 3.57, there exists $G \in \text{Gen}^+(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), p)$ such that $h[G] \in \text{Gen}^+(N, \mathcal{R}(\mathcal{F}, \mathcal{H}), h(p))$. By the assumption on \mathcal{H} and proposition 3.44, G has a common extension q such that $x_q = \bigcup_{\bar{p} \in G} x_{\bar{p}}$. Using proposition 3.44 again, $h[G]$ has a common extension q' such that

$$x_{q'} = \bigcup_{\bar{p} \in h[G]} x_{\bar{p}} = \bigcup_{\bar{p} \in G} h(x_{\bar{p}}) = \bigcup_{\bar{p} \in G} x_{\bar{p}} = x_q \quad (36)$$

by proposition 3.61. Therefore $(x_q, \mathcal{X}_q \cup \mathcal{X}_{q'})$ is a condition in $\mathcal{R}(\mathcal{F}, \mathcal{H})$ extending both of the filters G and $h[G]$. \square

3.7. Preservation of nonspecialness. The following definition is from [She98, Ch. IX, §4]. It was developed by Shelah for his proof that SH does not imply that all Aronszajn trees are special.

Definition 3.62. Let T be a tree of height ω_1 . A poset (P, \leq) is called (T, R) -preserving if for every countable $M \prec H_\kappa$, where κ is some sufficiently large regular cardinal, with $T, R, P \in M$ and $\delta_M \notin R$, every $p \in P \cap M$ has an (M, P) -generic extension q that is (M, P, T) -preserving, i.e. the following holds for all $x \in T_\delta$: if for all $A \subseteq T$ in M ,

$$x \in A \text{ implies } \exists y \in A \ y <_T x, \quad (37)$$

then for every P -name $\dot{A} \in M$ for a subset of T ,

$$q \Vdash \ulcorner x \in \dot{A} \text{ implies } \exists y \in \dot{A} \ y <_T x \urcorner. \quad (38)$$

In the case $R = \emptyset$, we just say that the poset is T -preserving; and when the poset is (T, R) -preserving for every ω_1 -tree T , we say that the poset is $(\omega_1\text{-tree}, R)$ -preserving.

The following lemma is straightforward.

Lemma 3.63. *If T is a Souslin tree, R is costationary and P is (T, R) -preserving, then T remains nonspecial in the forcing extension by P .*

In [Sch94] it is moreover shown that if T is a Souslin tree and $R \subseteq \omega_1$ is costationary, then T remains nonspecial in the forcing extension by a countable support iteration of (T, R) -preserving posets. In [AH07], the property itself is preserved:

Lemma 3.64 (Abraham). *Let $R \subseteq \omega_1$ be costationary and T an ω_1 -tree. Suppose $\vec{P} = (P_\xi, \dot{Q}_\xi : \xi < \mu)$ is a countable support iteration of length $\mu < \omega_2$ such that each iterand is (T, R) -preserving. Then P_μ is (T, R) -preserving.*

The preservation is proved for iterations of arbitrary length, but for a different type of iteration in [She98, Ch. IX].

It is here that we require the augmented game.

Lemma 3.65. *Let \mathcal{F} and \mathcal{H} be subfamilies of $[\theta]^{\leq \aleph_0}$ with \mathcal{F} closed under finite reductions. Let $R \subseteq \omega_1$. Suppose that $\psi \rightarrow \psi_{\min}$, and $\mathcal{E}(\omega_1 \setminus R)$ - ψ -globally, Extender does not have a winning strategy for $\mathfrak{D}_{\text{gen}}^*(\mathcal{F}, \mathcal{H})$. Then the forcing notion $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is $(\omega_1\text{-tree}, R)$ -preserving.*

Proof. Let $M \prec H_\kappa$ be a countable elementary submodel with $T, \mathcal{F}, \mathcal{H} \in M$, where T is some ω_1 -tree, and $\delta_M \notin R$. Let Z be the set of all $t \in T_{\delta_M}$ satisfying definition 3.62(37), i.e. for all $A \subseteq T$ in M , $t \in A$ implies $u <_T t$ for some $u \in A$. Then let (t_k, \dot{A}_k) ($k \in \mathbb{N}$) enumerate all pairs (t, \dot{A}) where $t \in Z$ and $\dot{A} \in M$ is an $\mathcal{R}(\mathcal{F}, \mathcal{H})$ -name for a subset of T . Given a condition $p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) \cap M$, we need to produce an $(M, \mathcal{R}(\mathcal{F}, \mathcal{H}))$ -generic extension that is moreover $(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), T)$ -preserving. We enumerate all of the dense subsets of $\mathcal{R}(\mathcal{F}, \mathcal{H})$ in M as $(D_k : k < \omega)$.

For each k and $\bar{p} \in \mathcal{R}(\mathcal{F}, \mathcal{H})$, a set $A_k^{\bar{p}} \subseteq T$ is defined as the collection of all $t \in T$, such that every cofinal $X \subseteq \mathcal{F}$, with $x_{\bar{p}} \subseteq z$ for all $z \in X$, has an extension $q \geq \bar{p}$ with

$$(39) \quad x_q \subseteq z \text{ for some } z \in X,$$

$$(40) \quad q \Vdash t \in \dot{A}_k.$$

Note that for each k , $\bar{p} \mapsto A_k^{\bar{p}}$ is definable in M .

ψ gives a $y \subseteq M$ such that $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$ (i.e. by equation (8)). Now the game $\mathfrak{D}_{\text{gen}}^*(M, y, \mathcal{F}, \mathcal{H}, p)$ is played. Let (p_k, X_k) denote Extender's k^{th} move. We shall describe a strategy for Extender. On even moves $2k$, Extender plays (p_{2k}, X_{2k}) such that

$$p_{2k} \in D_k, \tag{41}$$

by invoking corollary 3.22.2 using that fact $\psi \rightarrow \psi_{\min}$. We can implicitly use some well ordering so as to obtain a strategy.

After the $2k^{\text{th}}$ move has been played, we consider whether or not $t_k \in A_k^{p_{2k}}$. First suppose that it is. Then there exists $u_k <_T t_k$ in $A_k^{p_{2k}}$. Therefore, since $\psi \rightarrow \psi_{\min}$, we can apply corollary 3.22.1 with $Q = \{q : q \Vdash u_k \in \dot{A}_k\}$, so that equations (39) and (40) negate the second alternative (b), and thus Extender has a move (p_{2k+1}, X_{2k+1}) such that

$$p_{2k+1} \Vdash u_k \in \dot{A}_k. \tag{42}$$

Otherwise when $t_k \notin A_k^{p_{2k}}$, there exists a witness X_{2k+1} with $x_{p_{2k}} \subseteq z$ for all $z \in X_{2k+1}$ to the fact that there is no $q \geq p_{2k}$ satisfying (39) and (40) with $t := t_k$. Extender then makes the valid move (p_{2k+1}, X_{2k+1}) where $p_{2k+1} = p_{2k}$. Again we can use a well ordering to obtain an actual strategy.

Since Extender does not have a winning strategy in this game, there exists a sequence of plays by Complete such that Complete does not lose. And neither does Extender lose, because the described strategy is nonlosing by (41). Thus the game is drawn, and we can find a common extension $q \geq p_k$ for all k , satisfying $\{X_0, X_1, \dots\} \subseteq \mathcal{X}_q$. Since q is generic by proposition 3.12, it remains to verify that q is $(M, \mathcal{R}(\mathcal{F}, \mathcal{H}), T)$ -preserving.

Sublemma 3.65.1. *For all k , if $t_k \notin A_k^{p_{2k}}$ then $q \Vdash t_k \notin \dot{A}_k$.*

Proof. Fix k and suppose $t_k \notin A_k^{p_{2k}}$. Given $\bar{q} \geq q$ we need to prove that it does not force $t_k \in \dot{A}_k$. But since $X_{2k+1} \in \mathcal{X}_q$, there are cofinally many $z \in X_{2k+1}$ with $x_{\bar{q}} \subseteq z$. And by the choice of X_{2k+1} we cannot also have $\bar{q} \Vdash t_k \in \dot{A}_k$. \square

Take $t \in T_{\delta_M}$ satisfying equation (37), and an $\mathcal{R}(\mathcal{F}, \mathcal{H})$ -name $\dot{A} \in M$ for a subset of T . Then $(t, \dot{A}) = (t_k, \dot{A}_k)$ for some k . Supposing $\bar{q} \geq q$ forces $t_k \in \dot{A}_k$, then $t_k \in A_k^{p_{2k}}$ by the sublemma, and thus by (42) the verification is complete. \square

Let us also observe the following related technical fact.

Lemma 3.66. *σ -closed forcing notions are ω_1 -tree-preserving.*

Proof. This can be proved in a similar, but much simpler, manner to the proof of lemma 3.65. \square

We shall also require the following observation that is completely analogous in both its statement and its justification to lemma 3.53.

Lemma 3.67. *Let $R \subseteq \omega_1$ be costationary. If a forcing notion P is $(T, R \cup A)$ -preserving for some $A \in \text{NS}$, then P is (T, R) -preserving.*

We shall also need to preserve Aronszajn trees, i.e. make sure no uncountable branches are added. We use the following notion from [She98, Ch. IX, §4].

Definition 3.68. Let $h : \lim(\omega_1) \rightarrow \omega_1$ be a function whose domain is the countable limit ordinals. A tree T of height ω_1 is called *h -st-special* if there exists a function $f : \bigcup_{\alpha < \omega_1} T_{h(\alpha)} \rightarrow \omega_1$ satisfying

- (i) $x \in T_{h(\alpha)}$ implies $f(x) < \alpha$,
- (ii) for all $\alpha < \beta$, $x \in T_{h(\alpha)}$, $y \in T_{h(\beta)}$ and $x \leq_T y$ imply $f(x) \neq f(y)$.

It is easy to see that for any h , an h -st-special tree is not Souslin. The consequence we are interested in here is the following proposition, because h -st-specialness is obviously upwards absolute for \aleph_1 -preserving extensions.

Proposition 3.69. *If T is an h -st-special tree, then T has no uncountable branches.*

Lemma 3.70. *Let T be an Aronszajn tree. There is an ω_1 -tree-preserving forcing notion of cardinality \aleph_1 forcing that T is h -st-special for some h .*

Proof. We refer to Shelah's book. A slight modification—we prefer “ α ” to “ $\alpha \times \omega$ ”—of the forcing notion $Q(T)$ of definition 4.2 is shown to force T is h -st-special for some h in claim 4.4, and is shown to be ω_1 -tree-preserving in lemma 4.6. \square

3.8. Convex subfamilies of $(\mathcal{P}(\theta), \subseteq)$. The simplest class of families \mathcal{H} to be used as the second parameter are those that form convex subsets of $(\mathcal{P}(\theta), \subseteq)$, i.e. if $x \subseteq y$ are both in \mathcal{H} and $x \subseteq z \subseteq y$ then $z \in \mathcal{H}$.

Since \subseteq is a weaker relation than \sqsubseteq , such families \mathcal{H} are automatically convex in the initial segment ordering, and hence by corollary 3.42.1:

Proposition 3.71. *If \mathcal{H} is a convex subfamily of $(\mathcal{P}(\theta), \subseteq)$ then \mathcal{H} is upwards boundedly order closed.*

Convexity gives a relatively simple winning strategy for Complete in the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ when $y \in \mathcal{H}$.

Lemma 3.72. *Let \mathcal{F} be a directed subfamily of $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a λ -ideal, and \mathcal{H} be a convex subfamily of $(\mathcal{P}(\theta), \subseteq)$. Then Complete has a winning strategy for the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ whenever $y \in \mathcal{H}$ has cardinality $|y| < \lambda$ and $x_p \subseteq y$.*

Proof. We know that $\bigcup_{k < \omega} \mathcal{X}_{p_k}$ will be countable, and thus we can arrange a diagonalization $(Y_k : k < \omega)$ in advance, and since the \mathcal{X}_{p_k} 's will be increasing with k , we can also insist that $Y_k \in \mathcal{X}_{p_k}$ for all k . After Extender plays p_k on move k , we take care of some $Y_k \in \mathcal{X}_{p_k}$ according to the diagonalization.

$$Y'_k = \{x \in Y_k : x_{p_k} \subseteq x\} \text{ is } \subseteq^* \text{-cofinal in } \mathcal{F}, \quad (43)$$

and thus $Y''_k = \{x \in Y'_k : y \subseteq^* x\}$ is \subseteq^* -cofinal since we can assume that $y \in \downarrow \mathcal{F}$ by remark 3.2, and \mathcal{F} is \subseteq^* -directed. Now

$$Y''_k = \bigcup_{s \in \text{Fin}(y \setminus x_{p_k})} \{x \in Y'_k : y \setminus s \subseteq x\}, \quad (44)$$

and hence as $|y| < \lambda$, by the assumption on $\mathcal{J}(\mathcal{F})$ there exists $s_k \in \text{Fin}(y \setminus x_{p_k})$ such that

$$Y'''_k = \{x \in Y'_k : y \setminus s_k \subseteq x\} \text{ is } \subseteq^* \text{-cofinal.} \quad (45)$$

Complete plays s_k on move k . This describes the strategy for Complete.

And the end of the game, put $x_q = \bigcup_{k < \omega} x_{p_k}$ and $\mathcal{X}_q = \bigcup_{k < \omega} \mathcal{X}_{p_k}$. Then $x_q \in \mathcal{H}$ since $x_p, y \in \mathcal{H}$ and \mathcal{H} is convex; for every k , $x_q \subseteq x$ for all $x \in Y'''_k$ by proposition 3.14; and every $Y \in \bigcup_{k < \omega} \mathcal{X}_{p_k}$ appears as Y_k for some k , and thus $\{x \in Y : x_q \subseteq x\}$ is \subseteq^* -cofinal by (45). This proves that $q = (x_q, \mathcal{X}_q) \in \mathcal{R}(\mathcal{F}, \mathcal{H})$, and thus $q \geq p_k$ for all k and Complete wins the game. \square

Corollary 3.72.1. *Let \mathcal{F} be a directed subfamily of $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a λ -ideal, and \mathcal{H} be a convex subfamily of $(\mathcal{P}(\theta), \subseteq)$. Then Complete has a forward winning strategy for the game $\mathfrak{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$ whenever $y \in \mathcal{H}$ has cardinality $|y| < \lambda$ and $x_p \subseteq y$.*

Proof. We apply lemma 2.16 with $A = \text{Extender}$ and $X = \text{Complete}$ to obtain a forward winning strategy, and thus we need that Complete has a winning strategy, \mathcal{E} - F -globally, for some pair (\mathcal{E}, F) . Let $\mathcal{S} = \{V\}$ (cf. section 2.1), $\mathcal{T} = \mathcal{R}(\mathcal{F}, \mathcal{H})$ and let $F(V)$ be the function with domain \mathcal{T} where $F(V)(q) = \{x \in \mathcal{H} : |x| < \lambda \text{ and } x_q \subseteq x\}$. Then for every $q \in \mathcal{R}(\mathcal{F}, \mathcal{H})$ and every $x \in F(V)(q)$, Complete has a winning strategy in the game $\mathcal{D}_{\text{cmp}}(x, \mathcal{F}, \mathcal{H}, q)$ by lemma 3.72. Therefore, \mathcal{S} - F -globally, we have a winning strategy for Complete in the parameterized game $\mathcal{D}(V, x, q)$, where $\mathcal{D}(V, x, q) = \mathcal{D}_{\text{cmp}}(x, \mathcal{F}, \mathcal{H}, q)$.

Now suppose $P = (p_0, s_0), \dots, (p_k, s_k)$ is a position in the game $\mathcal{D}_{\text{cmp}}(x, \mathcal{F}, \mathcal{H}, q)$ with Extender to play. Then letting $x' = x \setminus \bigcup_{i=0}^k s_i$ and $q' = p_k$, clearly $x' \in F(V)(q')$ and $\mathcal{D}_{\text{cmp}}(x', \mathcal{F}, \mathcal{H}, q') = \mathcal{D}_{\text{cmp}}(x, \mathcal{F}, \mathcal{H}, q) \upharpoonright P$ is immediate from the rules of the game. Thus the hypothesis of lemma 2.16 is satisfied, and hence Complete has a forward winning strategy in the game $\mathcal{D}(V, x, q)$, \mathcal{S} - F -globally. In particular, Complete has a forward winning strategy in the game $\mathcal{D}_{\text{cmp}}(y, \mathcal{F}, \mathcal{H}, p)$, where y and p are from the hypothesis of the corollary. \square

Corollary 3.72.2. *Let \mathcal{F} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$ and let \mathcal{H} be convex. Suppose that ψ describes a suitability function such that $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$ implies $y \in \mathcal{H}$ and $x_p \subseteq y$. Then ψ -globally, Complete has a forward winning strategy in the game $\mathcal{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$.*

Proof. By corollary 3.72.1 with $\lambda = \aleph_1$. See also lemma 2.2. \square

Remark 3.73. Thus, assuming the hypothesis of corollary 3.72.2, ψ -globally, Complete has a nonlosing strategy in the game $\mathcal{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, by propositions 3.16 and 3.17.

We shall want to use corollary 3.72.2 with ψ satisfying $\psi \rightarrow \psi_{\min}$ in order to apply the theory of this section.

Definition 3.74. We let $\psi_{\text{cvx}}(M, y, \mathcal{F}, \mathcal{H}, p)$ be the conjunction of $\psi_{\min}(M, y, \mathcal{F}, \mathcal{H}, p)$ and $y \in \mathcal{H}$.

Remark 3.75. When both

- (1) \mathcal{F} is a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$,
- (2) \mathcal{H} is \subseteq^* -cofinal in \mathcal{F} ,

then ψ_{cvx} defines a suitability function for $(\mathcal{F}, \mathcal{H})$ fixed, as defined in definition 3.26, in that equation (10) holds. This is so because for any countable M , there exists $x \subseteq M$ in \mathcal{F} bounding $\mathcal{F} \cap M$ as \mathcal{F} is σ -directed and members of \mathcal{F} are countable, and hence there exists $y \in \mathcal{H}$ bounding $\mathcal{F} \cap M$.

Corollary 3.72.3. *Let \mathcal{F} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$ and let \mathcal{H} be convex. Then ψ_{cvx} -globally, Complete has a forward winning strategy in the game $\mathcal{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$.*

Proof. Corollary 3.72.2 and remark 3.75. \square

Note that since we are using ψ_{cvx} in the context of remark 3.75, we are not losing any generality here, in the crucial extendability property (definition 3.4), by requiring $\mathcal{F} = \mathcal{H}$.

Lemma 3.76. *Let \mathcal{H} be a directed subset of $(\mathcal{P}(\theta), \subseteq^*)$, with $\mathcal{J}(\mathcal{H}, \subseteq^*)$ a σ -ideal (e.g. when \mathcal{H} is σ -directed). Suppose θ has no countable decomposition into sets orthogonal to \mathcal{H} , and is the least ordinal for which this holds. Then \mathcal{H} is extendable.*

Proof. Take $x \in \mathcal{H}$ and let \mathcal{X} be a countable family of cofinal subsets of \mathcal{H} with $x \subseteq y$ for all $y \in X$ for all $X \in \mathcal{X}$. Applying lemma 2.4 with $\lambda = \aleph_1$, there exists $\alpha \geq \xi$ satisfying equation (4). Picking any $X \in \mathcal{X}$, find $y \in X$ such that $\alpha \in y$. Then $x \subseteq x \cup \{\alpha\} \subseteq y$ implies $x \cup \{\alpha\} \in \mathcal{H}$ by convexity, and thus $x \cup \{\alpha\}$ is a witness to extendability. \square

In the following three propositions (propositions 3.77–3.79) we are restricting \mathcal{T} only to include \mathcal{F} that are σ -directed subfamilies of $([\theta]^{\leq \aleph_0}, \subseteq^*)$.

Proposition 3.77. *ψ_{cvx} is provably coherent (cf. definition 3.47).*

Proof. The exact same proof as for proposition 3.48 works, because the element $y \cap y' \cap M$ defined there is in \mathcal{H} by convexity. \square

Proposition 3.78. *Provided we restrict ourselves to the class of convex \mathcal{H} , ψ_{cvx} provably respects \mathcal{D}_{gen} (cf. definition 3.49).*

Proof. Immediate from the definition and convexity. \square

Proposition 3.79. *Restricting to $\theta = \omega_1$, ψ_{cvx} provably respects isomorphisms (cf. definition 3.54).*

Proof. This follows from equation (32), by first observing that $h[y] = y$ and $h(x) = x$ for all $x \in \mathcal{F} \cap M$ by proposition 3.61. \square

3.9. Closed sets of ordinals. We shall only consider one more class of families \mathcal{H} in this paper, but there are certainly others of interest (see e.g. [Hir07c]).

Notation 3.80. For a family \mathcal{F} of subsets of some ordinal θ , and a subset $S \subseteq \theta$, typically stationary, we let $\mathcal{C}(\mathcal{F}, S)$ denote the family of all closed subsets of S (in the ordinal topology) in \mathcal{F} , i.e. $x \subseteq S$ is closed iff δ is a limit point of x and $\delta \in S$ imply $\delta \in x$.

Proposition 3.81. *$\mathcal{C}(\downarrow \mathcal{F}, S)$ is upwards boundedly order closed beyond S .*

Proof. Suppose $\mathcal{K} \subseteq \mathcal{C}(\downarrow \mathcal{F}, S)$ with $y \in \mathcal{C}(\downarrow \mathcal{F}, S)$ so that $x \sqsubseteq y$ for all $x \in \mathcal{K}$ and such that $\sup(\bigcup \mathcal{K}) \notin S$. Then $\bigcup \mathcal{K} \sqsubseteq y$, and in particular is in $\downarrow \mathcal{F}$ since y is, and $\bigcup \mathcal{K}$ is relatively closed in S because it is the union of a \sqsubseteq -chain of closed sets and by assumption, all of its limit points in S are strictly below its supremum. \square

Remark 3.82. In section 4 we are going to use the preceding proposition together with lemma 3.52 to show that under suitable assumptions, the poset $\mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$ is $\mathcal{E}(\omega_1 \setminus S)$ - α -proper. We remark that by standard counterexamples, it is not in general α -proper.

Lemma 3.83. *Let θ be an ordinal of uncountable cofinality. Let \mathcal{F} be a directed subfamily of $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a σ -ideal, and let $S \subseteq \theta$ be stationary. Suppose there is no stationary subset of S orthogonal to \mathcal{F} . Then $\mathcal{C}(\downarrow\mathcal{F}, S)$ is \mathcal{F} -extendable.*

Proof. Take $x \in \mathcal{C}(\downarrow\mathcal{F}, S)$, some countable family \mathcal{X} of cofinal subsets of \mathcal{F} with $x \subseteq y$ for all $y \in X$ for all $X \in \mathcal{X}$, and $\xi < \theta$. Since S is stationary, there exists a countable $M \prec H_\kappa$ with $x, \mathcal{F}, \mathcal{X}, \xi, S \in M$ and $\sup(\theta \cap M) \in S$. And by lemma 2.5, $\{y \in X : \sup(\theta \cap M) \in y\}$ is cofinal in \mathcal{F} for all $X \in \mathcal{X}$. Since $x \cup \{\sup(\theta \cap M)\} \subseteq S$ is also closed and in $\downarrow\mathcal{F}$, and since $\xi < \sup(\theta \cap M)$, $x \cup \{\sup(\theta \cap M)\}$ witnesses extendability. \square

Complete no longer has a winning strategy in the purely combinatorial game (as it did with \mathcal{H} convex in \subseteq), but is contented with a nonlosing strategy in $\mathfrak{D}_{\text{gen}}^*$, for some models.

Lemma 3.84. *Let \mathcal{F} be a directed subfamily $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a λ -ideal, and let $S \subseteq \theta$ be stationary. Suppose $M \prec H_{\theta^+}$ has cardinality $|M| < \text{cof}(\theta)$ with $\mathcal{F}, S \in M$,*

$$\sup(\theta \cap M) \notin S \quad (46)$$

and $p \in \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow\mathcal{F}, S)) \cap M$. Assuming that $\mathcal{C}(\downarrow\mathcal{F}, S)$ is \mathcal{F} -extendable, Complete has a nonlosing strategy for the game $\mathfrak{D}_{\text{gen}}^(M, y, \mathcal{F}, \mathcal{C}(\downarrow\mathcal{F}, S), p)$ whenever $y \subseteq M$ is in $\downarrow\mathcal{F}$ and of cardinality $|y| < \lambda$ with $x_p \subseteq y$.*

Proof. Set $\delta = \sup(\theta \cap M)$. We know that $\bigcup_{k < \omega} \mathcal{X}_{p_k}$ will be countable, and thus we can arrange a diagonalization in advance. After Extender plays its k^{th} move (p_k, X_k) , we take care of some $Y_k \in \mathcal{X}_{p_k}$ according to the diagonalization. Since $Y'_k = \{x \in Y_k : x_{p_k} \subseteq x\}$ is cofinal in \mathcal{F} , $Y''_k = \{x \in Y'_k : y \subseteq^* x\}$ is \subseteq^* -cofinal since $y \in \downarrow\mathcal{F}$ and \mathcal{F} is \subseteq^* -directed. Now $Y''_k = \bigcup_{s \in \text{Fin}(y \setminus x_{p_k})} \{x \in Y'_k : y \setminus s \subseteq x\}$, and hence by the assumption on $\mathcal{J}(\mathcal{F})$ there exists $s_k \in \text{Fin}(y \setminus x_{p_k})$ such that

$$Y'''_k = \{x \in Y'_k : y \setminus s_k \subseteq x\} \text{ is } \subseteq^*\text{-cofinal.} \quad (47)$$

Exactly the same argument shows that there exists $t_k \in \text{Fin}(y \setminus x_{p_k})$ such that

$$X'_k = \{x \in X_k : x_{p_k} \subseteq y \setminus t_k \subseteq x\} \text{ is } \subseteq^*\text{-cofinal.} \quad (48)$$

Complete plays $s_k \cup t_k$ on move k .

If Extender loses then Complete wins. Thus we may assume that Extender does not lose. Put $x_q = \bigcup_{k < \omega} x_{p_k}$ and $\mathcal{X}_q = \bigcup_{k < \omega} \mathcal{X}_{p_k} \cup \{X_0, X_1, \dots\}$. By proposition 3.6, for every $\xi < \theta$ in M , $\mathcal{D}_\xi \in M$ (cf. equation (14)) is dense, and thus $x_{p_k} \in \mathcal{D}_\xi$ for some k since Extender did not lose. Hence x_q is unbounded in δ , and therefore $x_q \in \mathcal{C}(\downarrow\mathcal{F}, S)$ for the same reason as in the proof of proposition 3.81. And for every k , by proposition 3.14, $x_q \subseteq x$ for all $x \in X'_k$ and all $x \in Y'''_k$ proving that $q = (x_q, \mathcal{X}_q) \in \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow\mathcal{F}, S))$ (see (48) and (47)). Thus the game is drawn. \square

When $\sup(\theta \cap M) \in S$, Complete no longer has a nonlosing strategy in the augmented game, but the unaugmented game is still OK.

Lemma 3.85. *Let \mathcal{F} be a directed subfamily of $(\mathcal{P}(\theta), \subseteq^*)$ with $\mathcal{J}(\mathcal{F}, \subseteq^*)$ a λ -ideal, and let $S \subseteq \theta$ be a stationary set. Suppose $M \prec H_{\theta^+}$ has cardinality $|M| < \text{cof}(\theta)$ with $\mathcal{F}, S \in M$, and $p \in \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S)) \cap M$. Assuming that there is no stationary subset of S orthogonal to \mathcal{F} , Complete has a nonlosing strategy for the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), p)$ whenever $y \subseteq M$ and $y \in \downarrow \mathcal{F}$ is of cardinality $|y| < \lambda$ with $x_p \subseteq y$.*

Proof. Set $\delta = \sup(\theta \cap M)$. The case $\delta \notin S$ has been dealt with in lemma 3.84, by lemma 3.83 and proposition 3.16. Assume then that $\delta \in S$. We know that $\bigcup_{k < \omega} \mathcal{X}_{p_k}$ will be countable, and thus we can arrange a diagonalization in advance. After Extender plays its k^{th} move p_k , we take care of some $Y_k \in \mathcal{X}_{p_k}$ according to the diagonalization. Since $\{x \in Y_k : x_{p_k} \subseteq x\} \in M$, and is cofinal in \mathcal{F} , by lemma 2.5,

$$Z_k = \{x \in Y_k : x_{p_k} \cup \{\delta\} \subseteq x\} \text{ is } \subseteq^*\text{-cofinal in } \mathcal{F}. \quad (49)$$

And thus $Z'_k = \{x \in Z_k : y \subseteq^* x\}$ is \subseteq^* -cofinal since $y \in \downarrow \mathcal{F}$ and \mathcal{F} is \subseteq^* -directed. Now $Z'_k = \bigcup_{s \in \text{Fin}(y \setminus x_{p_k})} \{x \in Z_k : y \setminus s \subseteq x\}$, and hence by the assumption on $\mathcal{J}(\mathcal{F})$ there exists $s_k \in \text{Fin}(y \setminus x_{p_k})$ such that

$$Z''_k = \{x \in Z_k : y \setminus s_k \subseteq x\} \text{ is } \subseteq^*\text{-cofinal}. \quad (50)$$

Complete plays s_k on move k . This describes the strategy for Complete.

At the end of the game, assume without loss of generality that Extender does not lose. Put

$$x_q = \bigcup_{k < \omega} x_{p_k} \cup \{\delta\}, \quad (51)$$

and $\mathcal{X}_q = \bigcup_{k < \omega} \mathcal{X}_{p_k}$. Every $Y \in \bigcup_{k < \omega} \mathcal{X}_{p_k}$ appears as Y_k for some k , and thus $\{x \in Y : x_q \subseteq x\}$ is \subseteq^* -cofinal by (50). Since $\mathcal{C}(\downarrow \mathcal{F}, S)$ is \mathcal{F} -extendable, $x_q \setminus \{\delta\}$ is unbounded in δ , which clearly implies that x_q is closed in S . Hence $x_q \in \mathcal{C}(\downarrow \mathcal{F}, S)$ as any of the sets x from (50) witnesses that $x_q \in \downarrow \mathcal{F}$. This proves that $q = (x_q, \mathcal{X}_q) \in \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$, and thus Complete does not lose the game. \square

Corollary 3.85.1. *Let \mathcal{F} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, and let $S \subseteq \theta$ be a stationary set. Suppose that there is no stationary subset of S orthogonal to \mathcal{F} . Let ψ be a formula describing a suitability function so that $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$ implies $y \in \downarrow \mathcal{F}$ and $x_p \subseteq y$. Then ψ -globally, Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$.*

Proof. Fix $M \in \mathcal{S}$ where \mathcal{S} comes from the global strategy associated with ψ , and suppose $p \in \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S)) \cap M$ and $\varphi(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), p, a_3)$. Then there exists $y \subseteq M$ satisfying $\psi(M, y, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), p)$. It remains to show that Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), p)$. Note that (by definition) $\mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S)) \in M$ and we may assume that S can be computed from this poset.

We proceed as in the proof of corollary 3.72.1, and thus we need a pair (\mathcal{E}, F) to be used with lemma 2.16 (of course different than (\mathcal{S}, ψ) above). Put $\mathcal{E} = \{M\}$, $\mathcal{T} = \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$ and let $F(M)$ be the function with domain $\mathcal{T} \cap M$

given by $F(M)(q) = \{x \subseteq M : x \in \downarrow \mathcal{F} \text{ and } x_q \subseteq x\}$. Then for every $q \in \mathcal{R}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S)) \cap M$ and every $x \in F(M)(q)$, Complete has a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(M, x, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q)$ by lemma 3.85. Therefore, there is a nonlosing strategy for Complete in the parameterized game $\mathfrak{D}(M, x, q)$, \mathcal{E} - F -globally, where $\mathfrak{D}(M, x, q) = \mathfrak{D}_{\text{gen}}(M, x, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{H}, S), q)$.

Supposing $P = (p_0, s_0), \dots, (p_k, s_k)$ is a position in the game $\mathfrak{D}_{\text{gen}}(M, x, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q)$ with Extender to play, if we put $x' = x \setminus \bigcup_{i=0}^k s_i$ and $q' = p_k$, clearly $x' \in F(M)(q')$ and $\mathfrak{D}_{\text{gen}}(M, x', \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q') = \mathfrak{D}_{\text{gen}}(M, x, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q) \upharpoonright P$ is immediate from the rules of the game. Thus the lemma 2.16 applies with $A = \text{Extender}$ and $X = \text{Complete}$, and hence Complete has a forward nonlosing strategy, \mathcal{E} - F -globally. In particular, Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(M, y, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), p)$, where y and p are the above fixed parameters. \square

Corollary 3.84.1. *Let \mathcal{F} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, and let $S \subseteq \theta$ be a stationary set. Suppose that there is no stationary subset of S orthogonal to \mathcal{F} . Let ψ be a formula describing a suitability function so that $\psi(M, y, \mathcal{F}, \mathcal{H}, p)$ implies $y \in \downarrow \mathcal{F}$ and $x_p \subseteq y$. Then $\mathcal{E}(\theta \setminus S, \theta)$ - ψ -globally, Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}^*(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$.*

Proof. This can be proved very similarly to corollary 3.85.1. The difference is that the game $\mathfrak{D}_{\text{gen}}^*$ is used in place of $\mathfrak{D}_{\text{gen}}$, but we only need to consider $M \in \mathcal{S}$ with $\sup(\theta \cap M) \notin S$. Thus we can invoke lemma 3.84 instead of lemma 3.85. At position $P = ((p_0, X_0), s_0), \dots, ((p_k, X_k), s_k)$ of the game $\mathfrak{D}_{\text{gen}}^*(M, x, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q)$, set $\bar{x} = x \setminus \bigcup_{i=0}^k s_i$ and $q' = p_k$. Then as in equation (48), there is a finite subset $z \subseteq \bar{x} \setminus x_{p_k}$ such that putting $x' = \bar{x} \setminus z$, $\{w \in X_i : x' \subseteq w\}$ is \subseteq^* -cofinal in \mathcal{F} for all $i = 0, \dots, k$. Since z is finite we can assume that z is \subseteq -minimal with this property. Now $\mathfrak{D}_{\text{gen}}^*(M, x', \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q')$ is ‘isomorphic’ to $\mathfrak{D}_{\text{gen}}^*(M, x, \mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S), q) \upharpoonright P$: By the minimality of z , the two games are identical for Extender; Complete has less room to move in the former game, but the difference has no effect on the outcome of the game. We will not formalize this argument further, and in any case corollary 3.84.1 is not applied in this paper. \square

Definition 3.86. We let $\psi_{\text{cls}}(M, y, \mathcal{F}, \mathcal{H}, p)$ be the conjunction of $\psi_{\text{min}}(M, y, \mathcal{F}, \mathcal{H}, p)$ and $y \in \downarrow \mathcal{F}$.

Remark 3.87. When \mathcal{F} is a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$ then we can make statements “ ψ_{cls} -globally” in that equation (10) holds.

Corollary 3.85.2. *Let \mathcal{F} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, and let $S \subseteq \theta$ be a stationary set with no stationary subset orthogonal to \mathcal{F} . Then ψ_{cls} -globally, Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$.*

Proof. Corollary 3.85.1 and remark 3.87. \square

Corollary 3.84.2. *Let \mathcal{F} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$, and let $S \subseteq \theta$ be a stationary set with no stationary subset orthogonal to \mathcal{F} . Then $\mathcal{E}(\theta \setminus S, \theta)$ - ψ_{cls} -globally, Complete has a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}^*(\mathcal{F}, \mathcal{C}(\downarrow \mathcal{F}, S))$.*

Proof. Corollary 3.84.1 and remark 3.87. \square

In the following three propositions (propositions 3.88–3.90) we are restricting \mathcal{T} only to include \mathcal{F} that are σ -directed subfamilies of $([\theta]^{\leq \aleph_0}, \subseteq^*)$.

Proposition 3.88. ψ_{cls} is provably coherent.

Proof. The exact same proof as for proposition 3.48 works, because the element $y \cap y' \cap M$ defined there is clearly in $\downarrow \mathcal{F}$. \square

Proposition 3.89. ψ_{cls} provably respects $\mathfrak{D}_{\text{gen}}$.

Proof. Immediate from the definition. \square

Proposition 3.90. Restricting to $\theta = \omega_1$, ψ_{cls} provably respects isomorphisms.

Proof. Exactly the same as for proposition 3.79. \square

4. HYBRID AND COMBINATORIAL PRINCIPIA

The following hybrid principium is the strongest statement of its type considered here.

(\ast_{max}) Let $(\mathcal{F}, \mathcal{H})$ be a pair of subfamilies of $[\theta]^{\leq \aleph_0}$ for some ordinal θ , with \mathcal{F} closed under finite reductions. If \mathcal{H} is \mathcal{F} -extendable and Extender has no winning strategy in the parameterized game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, ψ -globally for some $\psi \rightarrow \psi_{\text{min}}$, then there exists an uncountable $X \subseteq \theta$ such that every proper initial segment of X is in $\downarrow(\mathcal{H}, \sqsubseteq)$.

The corresponding combinatorial principium is as follows.

(\ast_{max}) Let $(\mathcal{F}, \mathcal{H})$ be a pair of subfamilies of $[\theta]^{\leq \aleph_0}$ for some ordinal θ , with \mathcal{F} closed under finite reductions. If \mathcal{H} is \mathcal{F} -extendable and Complete has a winning strategy in the parameterized game $\mathfrak{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$, ψ -globally for some $\psi \rightarrow \psi_{\text{min}}$, then there exists an uncountable $X \subseteq \theta$ such that every proper initial segment of X is in $\downarrow(\mathcal{H}, \sqsubseteq)$.

These principles are mentioned because they have enough constraints to be consistent.

Theorem 4.1. PFA implies (\ast_{max}).

Proof. Let \mathcal{F} and \mathcal{H} be as specified in the principle. By corollary 3.22.5, $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is completely proper. And by proposition 3.6, \mathcal{D}_ξ is dense for each $\xi < \theta$.

We still need a density argument to produce the desired uncountable $X \subseteq \theta$. For each $\alpha < \omega_1$, put

$$\mathcal{E}_\xi = \{p \in \mathcal{R}(\mathcal{F}, \mathcal{H}) : \text{otp}(x_p) > \xi\}. \quad (52)$$

Observe that each \mathcal{E}_ξ is dense: Given $\xi < \omega_1$ and $p \in \mathcal{R}(\mathcal{F}, \mathcal{H})$, take a countable elementary submodel $M \prec H_\kappa$ with $\xi, p, \mathcal{F}, \mathcal{H} \in M$. By complete properness, there exists $G \in \text{Gen}^+(M, P, p)$. Since the \mathcal{D}_ξ 's are dense,

$$M[G] \models \bigcup_{q \in G} x_q \text{ is cofinal in } \theta. \quad (53)$$

Then taking the transitive collapse, $\overline{M}[\overline{G}] \models \bigcup_{q \in \overline{G}} x_q$ is cofinal in $\overline{\theta}$. As $\xi \in M$ is fixed under the collapse, $\xi < \overline{\theta}$, and thus there exists $q \geq p$ in G with $\text{otp}(x_q) > \xi$. Hence $q \in \mathcal{E}_\xi$.

Now applying PFA to the proper poset $\mathcal{R}(\mathcal{F}, \mathcal{H})$, it has a filter G intersecting \mathcal{E}_ξ for all $\xi < \omega_1$. Therefore $X_G = \bigcup_{p \in G} x_p \subseteq \theta$ is uncountable. And every proper initial segment $y \sqsubset X$ is in $\downarrow(\mathcal{H}, \sqsubseteq)$, as required (proposition 3.8). \square

A slight formal weakening of these principles, namely requiring a *forward* strategy of Complete, allows us to significantly weaken PFA in the hypothesis. The hybrid version (\star) has already been presented in the introduction (page 8), and following is the corresponding combinatorial principle.

- (\star) Let $(\mathcal{F}, \mathcal{H})$ be a pair of subfamilies of $[\theta]^{\leq \aleph_0}$ for some ordinal θ , with \mathcal{F} closed under finite reductions. If \mathcal{H} is \mathcal{F} -extendable and Complete has a forward winning strategy in the parameterized game $\mathfrak{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$, ψ -globally for some $\psi \rightarrow \psi_{\min}$, then there exists an uncountable $X \subseteq \theta$ such that every proper initial segment of X is in $\downarrow(\mathcal{H}, \sqsubseteq)$.

Recall that \mathbb{D} -completeness implies complete properness (proposition 3.33).

Theorem 4.2. $\text{MA}(\mathbb{D}\text{-complete})$ *implies* (\star) .

Proof. The additional requirement that ψ -globally, Complete has a forward non-losing strategy in the game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, guarantees that $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is \mathbb{D} -complete by lemma 3.39 and remark 3.40. The rest of the proof is the same as the proof of theorem 4.1. \square

Just in case one decides to do a more in depth study of such principles, we might ask whether this weakening is purely formal.

Question 2. *Does either $(\star) \rightarrow (\star)_{\max}$ or $(\star) \rightarrow (\star)_{\max}$?*

Our goal here is to obtain principles compatible with CH. Although the “medicine” against (destroying) weak diamond has been taken for, e.g., (\star) , no “medicine” has been taken for the so called disjoint clubs (cf. [She00a]). Thus we would be surprised if (\star) is consistent with CH. On the other hand, we do not know of a counterexample.

Question 3. *Is (\star) compatible with CH?*⁹

The only way we know of to take the latter “medicine” is to make ‘geometrical’ restrictions on the the second family \mathcal{H} . First the hybrid principle:

- $(\star)_{\text{boc}}$ Let \mathcal{F} be a subfamily of $[\theta]^{\leq \aleph_0}$ for some ordinal θ , closed under finite reductions. Suppose \mathcal{H} is an upwards boundedly order closed subfamily of $([\theta]^{\leq \aleph_0}, \sqsubseteq)$. If \mathcal{H} is \mathcal{F} -extendable and Complete has a forward non-losing strategy in the parameterized game $\mathfrak{D}_{\text{gen}}(\mathcal{F}, \mathcal{H})$, ψ -globally for some $\psi \rightarrow \psi_{\min}$ that is coherent and respects $\mathfrak{D}_{\text{gen}}$, then there exists an uncountable $X \subseteq \theta$ such that every proper initial segment of X is in $\downarrow(\mathcal{H}, \sqsubseteq)$.

⁹ We did not use the word “consistent” because we want to avoid large cardinal considerations for the moment.

And the combinatorial principle:

- (\star_{boc}) Let \mathcal{F} be a subfamily of $[\theta]^{\leq \aleph_0}$ for some ordinal θ , closed under finite reductions. Suppose \mathcal{H} is an upwards boundedly order closed subfamily of $([\theta]^{\leq \aleph_0}, \sqsubseteq)$. If \mathcal{H} is \mathcal{F} -extendable and Complete has a forward winning strategy in the parameterized game $\mathfrak{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$, ψ -globally for some $\psi \rightarrow \psi_{\min}$ that is coherent and respects $\mathfrak{D}_{\text{gen}}$, then there exists an uncountable $X \subseteq \theta$ such that every proper initial segment of X is in $\downarrow(\mathcal{H}, \sqsubseteq)$.

Theorem 4.3. $\text{MA}(\alpha\text{-proper and } \mathbb{D}\text{-complete})$ *implies* (\star_{boc}).

Proof. The only thing that needs to be added to the proof of theorem 4.2 is that $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is α -proper. And this is by lemma 3.51 because \mathcal{H} is upwards boundedly order closed and by the additional requirements on ψ . \square

Corollary 4.3.1. (\star_{boc}) *is compatible with CH. More precisely, (\star_{boc}) is consistent with CH relative to a supercompact cardinal.*

Proof. Theorem 4.3 and Shelah's Corollary on page 24. \square

As we shall see below, (\star_{boc}) implies (\star) and thus has considerable large cardinal strength (cf. page 5). However, if we want to restrict to ω_1 then we expect that no large cardinals are necessary, as is the case with $(\star)_{\omega_1}$. We prove this for a slight weakening of (\star_{boc}).

- ($\star_{\text{boc}}^{\cong}$) This is the same principle as (\star_{boc}) except that we add the requirement that ψ respects isomorphisms.

The combinatorial principle ($\star_{\text{boc}}^{\cong}$) is exactly analogous.

Theorem 4.4. $\text{MA}(\alpha\text{-proper} + \mathbb{D}\text{-complete} + \text{pic} + \Delta_0\text{-}H_{\aleph_2}\text{-definable})$ *implies* ($\star_{\text{boc}}^{\cong}$) $_{\omega_1}$.

Proof. Letting \mathcal{F} and \mathcal{H} be subsets of $[\omega_1]^{\leq \aleph_0}$ satisfying the requirements of the principle, we need in addition to the proof of theorem 4.3 to show that $\mathcal{R}(\mathcal{F}, \mathcal{H})$ satisfies the properness isomorphism condition and is Δ_0 -definable over H_{\aleph_2} . These are true by corollary 3.56.1 and example 3.59, respectively. \square

Corollary 4.4.1. ($\star_{\text{boc}}^{\cong}$) $_{\omega_1}$ *is consistent with CH relative to the consistency of ZFC.*

Proof. By theorem 3.1. \square

This raises an interesting question. “Naturally occurring” combinatorial principles on ω_1 generally (always?) have no large cardinal strength. The only explanation that the author knows of is that they can be forced without collapsing cardinals. Perhaps there is a more satisfactory explanation? (It is quite possible that there is a well known explanation that the author is simply unaware of.) If all naturally occurring combinatorial principles on ω_1 —notice that this notion has not been clearly defined, and this may well be the essential point—can be decided without large cardinals, then we should be able to prove the consistency of $(\star_{\text{max}})_{\omega_1}$ without large cardinal assumptions.

Question 4. *Is $(\star_{\max})_{\omega_1}$ consistent relative to ZFC?*

Even if question 4 has a positive answer, it is conceivable that adding CH requires large cardinals.

Question 5. *Is the conjunction of $(\star_{\text{boc}})_{\omega_1}$ and CH relatively consistent with ZFC? If question 3 has a positive answer, is $(\star)_{\omega_1}$ and CH relatively consistent with ZFC?*

Note that we can obtain the consistency of $(\star_{\max})_{\omega_1}$ or $(\star_{\text{boc}})_{\omega_1} + \text{CH}$ by assuming the existence of an inaccessible cardinal. E.g. one can produce a model of $(\star_{\text{boc}})_{\omega_1}$ and CH by iterating up to an inaccessible cardinal instead of using the properness isomorphism property to guarantee a suitable chain condition.

Now we observe that the combinatorial version (\star_{boc}) is already weak enough to be consistent with the existence of a nonspecial Aronszajn tree.

Theorem 4.5. *The conjunction of (\star_{boc}) , CH and the existence of a nonspecial Aronszajn tree is consistent relative to a supercompact cardinal.*

Proof. By the argument in the proof of theorem 4.1, to obtain a model of (\star_{boc}) it suffices to ensure that for every pair $(\mathcal{F}, \mathcal{H})$ as specified in the principle, for every family of size \aleph_1 consisting of dense subsets of the poset $\mathcal{R}(\mathcal{F}, \mathcal{H})$, there exists a filter intersecting every member of the family, i.e. we prove $\text{MA}(\mathcal{R}(\mathcal{F}, \mathcal{H}))$ for every such $(\mathcal{F}, \mathcal{H})$.

Using the argumentation of the proof of the consistency of PFA (see e.g. [FMS88, §1]), we can obtain such a model by extending by a countable support iteration $(P_\xi, \dot{Q}_\xi : \xi < \kappa)$ where κ is a supercompact cardinal and each iterand \dot{Q}_ξ is either a P_ξ -name for a poset of the form $\mathcal{R}(\dot{\mathcal{F}}_\xi, \dot{\mathcal{H}}_\xi)$ with $(\dot{\mathcal{F}}_\xi, \dot{\mathcal{H}}_\xi)$ as above, or else a P_ξ -name for a Lévy collapse of the form $\text{Col}(\aleph_1, \theta)$ which in particular is a σ -closed poset. As argued in the proof of theorem 4.3, each \dot{Q}_ξ is α -proper, and in the case $\dot{Q}_\xi = \mathcal{R}(\dot{\mathcal{F}}_\xi, \dot{\mathcal{H}}_\xi)$, \mathcal{R} has the σ -complete completeness system from remark 3.40, and the collapsing poset has the trivial completeness system (i.e. $\text{Gen}(M, P) = \text{Gen}^+(M, P)$ whenever P is σ -closed). Therefore, by Shelah's Theorem on page 24, the iteration does not add new reals and thus if the ground model satisfies CH then so does the extension.

Now it is specified (in particular) that ψ -globally, Complete has a winning strategy in the game $\mathfrak{D}_{\text{cmp}}(\mathcal{F}, \mathcal{H})$. Hence Complete has, ψ -globally, a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}^*(\mathcal{F}, \mathcal{H})$ by proposition 3.17. Thus by lemma 3.65, the poset $\mathcal{R}(\mathcal{F}, \mathcal{H})$ is ω_1 -tree-preserving. And σ -closed posets are also ω_1 -tree-preserving by lemma 3.66. Therefore, every iterand of our iteration is ω_1 -tree-preserving, and thus $\varinjlim_{\xi < \kappa} P_\xi$ is ω_1 -tree-preserving by Schlindwein's theorem [Sch94] discussed on page 34. Thus by lemma 3.63, if we begin with a ground model satisfying CH containing a Souslin tree T then our extension satisfies (\star_{boc}) , CH and T remains nonspecial. To ensure that T moreover remains Aronszajn in our extension, we can further assume that T is h -st-special in the ground model (this is better explained in the proof of theorem 4.7). \square

Theorem 4.6. *The conjunction of $(\star_{\text{boc}}^{\cong})_{\omega_1}$, CH and the existence of a nonspecial Aronszajn tree is consistent relative to ZFC.*

Proof. This is done very similarly to the proof of theorem 3.1, except that we only include iterands of the form $\mathcal{R}(\mathcal{F}, \mathcal{H})$, where $(\mathcal{F}, \mathcal{H})$ names a pair of subfamilies of $[\omega_1]^{\leq \aleph_0}$ as in the specification of $(\star_{\text{boc}}^{\cong})$. Thus, as in the preceding theorem, each iterand is ω_1 -tree-preserving in addition to being α -proper and has a fixed σ -complete completeness system and is in pic. Therefore, by starting out with a ground model containing an h -st-special Souslin tree and satisfying GCH, we end up with a model satisfying $(\star_{\text{boc}}^{\cong})$, CH and the existence of a nonspecial Aronszajn tree. \square

Lemma 4.1. *(\star_{boc}) implies $(*)$.*

Proof. Let \mathcal{H} be a σ -directed subfamily of $([\theta]^{\leq \aleph_0}, \subseteq^*)$. We assume that the second alternative (2) of $(*)$ fails, and prove that the first alternative (1) holds. Next we verify that the pair $(\downarrow \mathcal{H}, \downarrow \mathcal{H})$ satisfies the hypotheses of the principle (\star_{boc}) . Since $\downarrow \mathcal{H}$ is convex, it is upwards boundedly order closed by proposition 3.71. By restricting to a smaller ordinal if necessary, we can assume that θ is least ordinal that has no countable decomposition into pieces orthogonal to \mathcal{H} . Then by lemma 3.76, $\downarrow \mathcal{H}$ is extendable (i.e. $\downarrow \mathcal{H}$ is $\downarrow \mathcal{H}$ -extendable). We use ψ_{cvx} from definition 3.74. Complete has a forward winning strategy in the game $\mathcal{D}_{\text{cmp}}(\downarrow \mathcal{H}, \downarrow \mathcal{H})$, ψ_{cvx} -globally, by corollary 3.72.2. And $\psi_{\text{cvx}} \rightarrow \psi_{\text{min}}$ is coherent and respects \mathcal{D}_{gen} by propositions 3.77 and 3.78. Therefore, (\star_{boc}) gives an uncountable $X \subseteq \theta$ with every proper initial segment in $\downarrow \mathcal{H}$. In particular, X is locally in $\downarrow \mathcal{H}$ establishing the first alternative. \square

Lemma 4.2. *$(\star_{\text{boc}}^{\cong})_{\omega_1}$ implies $(*)_{\omega_1}$.*

Proof. Additionally to the proof of lemma 4.1, restricting $\mathcal{F}, \mathcal{H} \subseteq [\omega_1]^{\leq \aleph_0}$ implies that ψ_{cvx} respects isomorphisms by proposition 3.79. \square

Now we have answered the Abraham–Todorćević question (cf. page 5), by establishing that $(*)$ does not imply that all Aronszajn trees are special (cf. theorem 1.1).

Proof of theorem 1.1. Theorem 4.5 and lemma 4.1 for the unrestricted principle, and theorem 4.6 and lemma 4.2 for $(*)_{\omega_1}$. \square

Theorem 4.7. *The conjunction of $(\star_s)_{\omega_1}$ and $(*)_{\omega_1}$ is consistent with CH and the existence of a nonspecial Aronszajn tree relative to ZFC.*

Proof. First we describe the ground model V . Assume, by going to a forcing extension if necessary, that CH holds and $2^{\aleph_1} = \aleph_2$. By further forcing if necessary, we may assume moreover that the stationary coideal NS^+ does not satisfy the \aleph_2 -chain condition (see e.g. [AS02]). Thus we can fix a maximal antichain $\mathcal{A} \subseteq \text{NS}^+$ of cardinality \aleph_2 (maximality is unimportant here). By forcing yet again if necessary, we may assume that in addition there exists a Souslin tree T .

Next we let W denote the forcing extension of V by the ω_1 -tree-preserving (and thus proper) forcing notion from lemma 3.70, so that in W , T is h -st-special for some h .

In W : We construct a forcing notion $P = \varinjlim_{\xi < \omega_2} P_\xi$ using an iterated forcing construction $(P_\xi, \dot{Q}_\xi : \xi < \omega_2)$ of length ω_2 with countable supports and inverse limits. Just as in the proof of theorem 3.1, and as is standard, each iterand \dot{Q}_ξ is forced to be of size at most \aleph_2 , and satisfy the pic. Thus, as each P_ξ has the the \aleph_2 -cc and a dense suborder of cardinality at most \aleph_2 , we can use standard bookkeeping to obtain an enumeration $(\dot{R}_\xi, \dot{\mathcal{H}}_\xi : \xi < \omega_2)$ in advance such that, viewing P_ξ -names as P_η -names for $\xi \leq \eta$, every pair of P_ξ -names $(\dot{R}, \dot{\mathcal{H}})$ for a subset \dot{R} of ω_1 and a subfamily $\dot{\mathcal{H}}$ of $[\omega_1]^{\leq \aleph_0}$, respectively, appears as $(\dot{R}_\eta, \dot{\mathcal{H}}_\eta)$ for cofinally many $\xi \leq \eta < \omega_2$. By skipping steps, we may assume that for all ξ ,

- (i) $P_\xi \Vdash \dot{R}_\xi \subseteq \omega_1$ is stationary,
- (ii) $P_\xi \Vdash \dot{\mathcal{H}}_\xi$ is a σ -directed subset of $([\omega_1]^{\leq \aleph_0}, \subseteq^*)$.

We also recursively choose P_ξ -names \dot{S}_ξ for subsets of ω_1 , according to which one of the following mutually exclusive cases holds:

- (a) $P_\xi \Vdash \dot{R}_\xi = \dot{O}_\xi$ for some $O_\xi \in \mathcal{A}$,
- (b) $p \Vdash \dot{R}_\xi \cap S \in \text{NS}$ for all $S \in \mathcal{A} \cup \{\dot{S}_\gamma : \gamma < \xi\}$ for some $p \in P_\xi$,
- (c) $p \Vdash \dot{R}_\xi \in \{\dot{S}_\gamma : \gamma < \xi\} \setminus \mathcal{A}$ for some $p \in P_\xi$ and case (b) fails,
- (d) otherwise.

In case (a), we set $\dot{S}_\xi = \dot{O}_\xi$; in case (b), we define \dot{S}_ξ so that $p \Vdash \dot{S}_\xi = \dot{R}_\xi$ whenever $p \in P_\xi$ is as specified there, and $q \Vdash \dot{S}_\xi = \emptyset$ for q incompatible with such a p ; in case (c), we define \dot{S}_ξ so that $p \Vdash \dot{S}_\xi = \dot{R}_\xi$ whenever $p \in P_\xi$ is as specified in (c), and $q \Vdash \dot{S}_\xi = \emptyset$ if q is incompatible with any such p ; in case (d), we set $\dot{S}_\xi = \emptyset$. Thus we have that the nonempty members of $\mathcal{A} \cup \{\dot{S}_\gamma : \gamma \leq \xi\}$ are forced to form an antichain of NS^+ . Put $\Gamma_\xi = \{\gamma \leq \xi : P_\gamma \Vdash \dot{S}_\gamma = \dot{O}_\gamma\}$. Then observe that

$$P_\gamma \Vdash \dot{S} \cap \dot{S}_\gamma \in \text{NS} \quad \text{for all } S \in \mathcal{A} \setminus \Gamma_\xi \text{ and all } \gamma \leq \xi. \quad (54)$$

We shall also ensure that at each stage ξ ,

$$P_\xi \Vdash \dot{Q}_\xi \text{ is } \mathcal{E}(\omega_1 \setminus \dot{S}_\xi)\text{-}\alpha\text{-proper} \quad (55)$$

$$P_\xi \Vdash \dot{Q}_\xi \text{ is } (\omega_1\text{-tree}, \dot{S}_\xi)\text{-preserving} \quad (56)$$

We now start working in the forcing extension by P_ξ , in order to specify \dot{Q}_ξ . Let \mathcal{H}_ξ and \mathcal{S}_ξ be the interpretations of $\dot{\mathcal{H}}_\xi$ and \dot{S}_ξ , respectively. If $\mathcal{S}_\xi = \emptyset$ we let \dot{Q}_ξ be trivial. Otherwise, we now specify \dot{Q}_ξ , determined in order by the following cases.

Case 1. \mathcal{H}_ξ has a stationary orthogonal set, but ω_1 has no countable decomposition into sets orthogonal to \mathcal{H}_ξ . Here we force with $\dot{Q}_\xi = \mathcal{R}(\downarrow \mathcal{H}_\xi)$.

First we verify that it forces the desired object. Let $X_{\dot{G}}$ be a \dot{Q}_ξ -name for the generic object. Clearly it is forced to be locally in $\downarrow \mathcal{H}_\xi$ (cf. proposition 3.8). Since ω_1 is in fact the least ordinal that cannot be decomposed into countable many

pieces orthogonal to \mathcal{H}_ξ (proposition 2.3), the fact that \mathcal{H}_ξ is σ -directed implies that it is extendable by lemma 3.76. Therefore $X_{\dot{G}}$ is forced to be uncountable (proposition 3.6).

By corollary 3.72.3, ψ_{cvx} -globally, Complete has a forward winning strategy in the game $\mathfrak{D}_{\text{cmp}}(\downarrow \mathcal{H}_\xi, \downarrow \mathcal{H}_\xi)$. Now $\downarrow \mathcal{H}_\xi$ is upwards boundedly order closed (proposition 3.71), $\psi_{\text{cvx}} \rightarrow \psi_{\text{min}}$ is coherent and respects $\mathfrak{D}_{\text{gen}}$ (propositions 3.77 and 3.78) and ψ_{cvx} -globally, Complete has a forward nonlosing strategy for $\mathfrak{D}_{\text{gen}}(\downarrow \mathcal{H}_\xi, \downarrow \mathcal{H}_\xi)$ (propositions 3.16 and 3.17). Therefore, Q_ξ is in \mathbb{D} -complete by lemma 3.39 as witnessed by the fixed pair of formulae given in remark 3.40. And Q_ξ is α -proper by lemma 3.51, and in particular (55) is satisfied. Also, since ψ_{cvx} respects isomorphisms (proposition 3.79), Q_ξ satisfies the properness isomorphism condition by corollary 3.56.1.

We verify that Q_ξ is ω_1 -tree-preserving. Indeed by proposition 3.17, ψ_{cvx} -globally, Complete has a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}^*(\downarrow \mathcal{H}_\xi, \downarrow \mathcal{H}_\xi)$; hence, we obtain the ω_1 -tree-preserving property from lemma 3.65.

Case 2. \mathcal{H}_ξ has no stationary set orthogonal to it. In this case we force with $Q_\xi = \mathcal{R}(\downarrow \mathcal{H}_\xi, \mathcal{C}(\downarrow \mathcal{H}_\xi, S_\xi))$.

By lemma 3.83, $\mathcal{C}(\downarrow \mathcal{H}_\xi, S_\xi)$ is \mathcal{H}_ξ -extendable, and thus $X_{\dot{G}}$ is forced to be uncountable. And Q_ξ forces that every proper initial segment of $X_{\dot{G}}$ is in $\downarrow(\mathcal{C}(\downarrow \mathcal{H}_\xi, S_\xi), \sqsubseteq)$ by proposition 3.8.

By corollary 3.85.2, ψ_{cls} -globally, Complete has a forward nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}(\downarrow \mathcal{H}_\xi, \mathcal{C}(\downarrow \mathcal{H}_\xi, S_\xi))$. Thus Q_ξ is in \mathbb{D} -complete by lemma 3.39. Since $\mathcal{C}(\downarrow \mathcal{H}_\xi, S_\xi)$ is upwards boundedly order closed beyond S_ξ (proposition 3.81) and $\psi_{\text{cls}} \rightarrow \psi_{\text{min}}$ is coherent and respects $\mathfrak{D}_{\text{gen}}$ (propositions 3.88 and 3.89), Complete's nonlosing strategy ψ -globally ensures that Q_ξ is $\mathcal{E}(\omega_1 \setminus S_\xi)$ - α -proper by lemma 3.52, and thus (55) holds. And since ψ_{cls} respects isomorphisms (proposition 3.90), Q_ξ is in pic by corollary 3.56.1.

As for equation (56), Complete has a nonlosing strategy in the game $\mathfrak{D}_{\text{gen}}^*(\downarrow \mathcal{H}_\xi, \mathcal{C}(\downarrow \mathcal{H}_\xi, S_\xi))$, $\mathcal{E}(\omega_1 \setminus S_\xi)$ - ψ_{cls} -globally, by corollary 3.84.2. Hence Q_ξ is $(\omega_1$ -tree, S_ξ)-preserving by lemma 3.65.

Case 3. There is a countable decomposition of ω_1 into sets orthogonal to \mathcal{H}_ξ . Then Q_ξ is trivial.

Having defined the iteration, we verify that it has the desired properties. In the final forcing extension $W[G]$ of W by P : We know that the nonempty members \mathcal{B} of $\mathcal{A} \cup \{S_\xi : \xi < \omega_2\}$ form an antichain of NS^+ . And \mathcal{B} is in fact a maximal antichain. For suppose to the contrary that there is a stationary set R with $R \cap S \in \text{NS}$ for all $S \in \mathcal{B}$. Since P is proper and thus does not collapse \aleph_1 and since it has the \aleph_2 -cc, R appears at some intermediate stage, i.e. there exists $\eta < \omega_2$ such that $R \in W[G_\eta]$, where $G_\eta = G \restriction P_\eta$ is a generic filter on P_η . Therefore by our bookkeeping, we can find $\xi < \omega_2$ such that $\dot{R}_\xi[G] = R$, and thus p forces $\dot{R}_\xi \cap S \in \text{NS}$ for all $S \in \mathcal{A} \cup \{\dot{S}_\gamma : \gamma < \xi\}$, for some $p \in G_\xi$. Thus we are in case (b), and hence $p \Vdash \dot{S}_\xi = \dot{R}_\xi$, and we arrive at the contradiction $R \in \mathcal{B}$.

Let us verify that \mathcal{B} serves to instantiate $(*)_s$. Suppose \mathcal{H} is a σ -directed subfamily of $([\omega_1]^{\leq \aleph_0}, \subseteq^*)$, and that the second alternative (2) of $(*)_s$ fails, and

hence there is no stationary set orthogonal to \mathcal{H} . Take $S \in \mathcal{B}$. Now there exists $\xi < \omega_2$ such that

$$(\mathcal{H}, S) = (\dot{\mathcal{H}}_\xi[G], \dot{R}_\xi[G]). \quad (57)$$

If $S \in \mathcal{A}$, then $p \Vdash \dot{R}_\xi = \dot{S}$ for some $p \in G$, and hence by our bookkeeping, we can find such an ξ so that we are furthermore in case (a).

Otherwise, $S \in \{S_\gamma : \gamma < \omega_2\} \setminus \mathcal{A}$, and thus $p \Vdash \dot{R}_\xi = \dot{S}_\gamma \notin \mathcal{A}$ for some $\gamma < \omega_2$ and $p \in G_\xi$. By our bookkeeping, we can find such an ξ so that $\xi > \gamma$ and so that moreover we are not in case (b). Therefore, we are in case (c).

In either of these two situations, $p \Vdash \dot{S}_\xi = \dot{R}_\xi$ for some $p \in G$. And we arrive at case 2 of the construction of \dot{Q}_ξ . Hence $Q_\xi = \dot{Q}_\xi[G]$ forces an uncountable $X \subseteq \omega_1$ with all of its initial segments in $\downarrow(\mathcal{C}(\downarrow\mathcal{H}, S), \sqsubseteq)$. In particular, X relatively closed in S and is locally in $\downarrow\mathcal{H}$, as wanted.

The verification that $(*)_{\omega_1}$ holds is similar. Here we will arrive in either case 1 or 2 depending on whether there is a stationary set orthogonal to \mathcal{H} , and in either case we obtain an uncountable $X \subseteq \omega_1$ locally in $\downarrow\mathcal{H}$.

Next we prove that $W[G] \models \text{CH}$, by showing that P does not add any new reals. Suppose towards a contradiction that there is a real number $r \in \mathbb{R}$ (in $W[G]$) that is not in the model W . Then at some intermediate stage η , $r \in W[G_\eta]$, and thus the initial segment P_η of the iteration has added a new real. Since $|\mathcal{A}| = \aleph_2$ while $|\Gamma_\eta| \leq \aleph_1$, there exists $S \in \mathcal{A} \setminus \Gamma_\eta$. But for all $\xi < \eta$, $P_\xi \Vdash \dot{Q}_\xi$ is $\mathcal{E}(S \setminus \dot{S}_\xi)$ - α -proper, by equation (55), and $P_\xi \Vdash \dot{S} \cap \dot{S}_\xi \in \text{NS}$ by equation (54). Therefore, by lemma 3.53, we in fact have that $P_\xi \Vdash \dot{Q}_\xi$ is $\mathcal{E}(S)$ - α -proper. Since we have also demonstrated that each iterand is in \mathbb{D} -complete, by Shelah's Theorem on page 24, P_η does not add new reals, a contradiction.

To conclude the proof, we need to show that the Souslin tree T in the ground model V , is a nonspecial Aronszajn tree in the forcing extension $W[G]$. Since $W \models \ulcorner T \text{ is } h\text{-st-special} \urcorner$, we know that T is Aronszajn by proposition 3.69. Hence it remains to establish nonspecialness. However, supposing towards a contradiction that it is special, the specializing function appears in some intermediate model, and thus $W[G_\eta] \models \ulcorner T \text{ is special} \urcorner$ for some $\eta < \omega_2$.

By equation (56), $P_\xi \Vdash \dot{Q}_\xi$ is (T, \dot{S}_ξ) -preserving, for all $\xi < \eta$. And taking any $S \in \mathcal{A} \setminus \Gamma_\eta$, we have $P_\xi \Vdash S \cap \dot{S}_\xi \in \text{NS}$ for all $\xi < \eta$ by (54). Therefore, $P_\xi \Vdash (\omega_1 \setminus S) \cup \dot{S}_\xi = (\omega_1 \setminus S) \cup \dot{A}_\xi$, where $P_\xi \Vdash \dot{A}_\xi \in \text{NS}$. Since every P_ξ forces that \dot{Q}_ξ is $(T, (\omega_1 \setminus S) \cup \dot{S}_\xi)$ -preserving, by lemma 3.67, $P_\xi \Vdash \dot{Q}_\xi$ is $(T, (\omega_1 \setminus S))$ -preserving. But then P_η is $(T, (\omega_1 \setminus S))$ -preserving by lemma 3.64. And therefore, since W is an ω_1 -tree-preserving forcing extension of V , $W[G_\eta]$ is also a $(T, (\omega_1 \setminus S))$ -preserving extension of V . Hence, by lemma 3.63, $W[G_\eta] \models \ulcorner \Vdash T \text{ is nonspecial} \urcorner$, a contradiction. \square

Question 6. *Is the conjunction of $(*_s)$ and $(*)$ consistent with CH and the existence of a nonspecial Aronszajn tree relative to a supercompact cardinal?*

Remark 4.3. We expect that the proof of theorem 4.7 can be readily generalized to prove the consistency of $(*_s)$ with CH; indeed, this should probably be stated and proved as a separate theorem. In the proposed proof one would construct

maximal antichains in $(\text{NS}_\theta^+, \subseteq)$ for each regular $\theta < \kappa$ simultaneously, where κ is a supercompact cardinal. Then at any intermediate stage $\xi < \kappa$ of the iteration, we would have that P_ξ is $\mathcal{E}(S, \theta)$ - α -proper for some regular cardinal $\theta < \kappa$ and some stationary $S \subseteq \theta$; hence, the iteration would not add new reals. Forcing $(*)$ to hold simultaneously should not pose any difficulties.

However, preserving the nonspecialness of some Aronszajn tree does seem problematic, and seems to require a new approach even if question 6 has a positive answer. The difficulty is that if we try to preserve the (T, R) -preserving property for some costationary $R \subseteq \omega_1$, we run out of possibilities for R after 2^{\aleph_1} many steps; unlike for $\mathcal{E}(S)$ - α -properness, we cannot use (T, R) -preserving with $R \subseteq \theta > \omega_1$.

5. APPLICATIONS

5.1. The principle (A).

Definition 5.1. We say that a lower subset \mathcal{L} of some poset (P, \leq) is λ -generated if there exists a subset $\mathcal{G} \subseteq \mathcal{L}$ of cardinality at most λ such that every member x of \mathcal{L} satisfies $x \leq y$ for some member y of \mathcal{G} , i.e. $\mathcal{L} \subseteq \downarrow(\mathcal{G}, \leq)$.

We say that \mathcal{L} is *locally λ -generated* if for every $a \in P$, $\downarrow\mathcal{L} \cap \downarrow a$ is λ -generated.

Proposition 5.2. *Let λ be an infinite cardinal. An ideal is λ -generated in the standard sense iff it is λ -generated as a lower set.*

We shall consider lower subsets of posets of the form $([\theta]^{\leq \aleph_0}, \subseteq)$. For a family \mathcal{F} of sets and a set A , write $\mathcal{F} \cap A = \{x \cap A : x \in \mathcal{F}\}$.

Proposition 5.3. *Let $\mathcal{H} \subseteq [\theta]^{\leq \aleph_0}$. Then $\downarrow\mathcal{H}$ is locally λ -generated in $([\theta]^{\leq \aleph_0}, \subseteq)$ iff $\downarrow\mathcal{H} \cap A$ is λ -generated for every countable $A \subseteq \theta$.*

The next lemma generalizes [EN07, Theorem 2.6]. We refer to two of the standard cardinal characteristics of the continuum \mathfrak{p} and \mathfrak{b} (see e.g. [Bla03]). \mathfrak{p} is the smallest cardinality of a subfamily $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ with the *finite intersection property*, meaning that $\bigcap F$ is infinite for every nonempty finite $F \subseteq \mathcal{A}$, and such that \mathcal{F} has no infinite \subseteq^* -lower bound.

Lemma 5.4 ($\mathfrak{p} > \lambda$). *Let \mathcal{H} be a directed subset of $([\theta]^{\leq \aleph_0}, \subseteq^*)$. If $\downarrow\mathcal{H}$ is locally λ -generated then*

$$\mathcal{H}^{\perp\perp} = \downarrow(\mathcal{H}, \subseteq^*). \quad (58)$$

Proof. We need to show that $\mathcal{H}^{\perp\perp} \subseteq \downarrow(\mathcal{H}, \subseteq^*)$, because the opposite inclusion holds for any family \mathcal{H} . Suppose $y \in [\theta]^{\leq \aleph_0}$ is not in $\downarrow(\mathcal{H}, \subseteq^*)$. By assumption, $\downarrow\mathcal{H} \cap y$ is λ -generated, say by some family of generators $\mathcal{G} \subseteq \mathcal{H} \cap y$ with $|\mathcal{G}| \leq \lambda$. Since \mathcal{H} is directed, there can be no finite $F \subseteq \mathcal{G}$ with $y \subseteq^* \bigcup F$. Therefore, $y \setminus \mathcal{G} = \{y \setminus x : x \in \mathcal{G}\}$ is a family of cardinality at most λ with the finite intersection property. Thus $\mathfrak{p} > \lambda$ yields an infinite $z \subseteq y$ that is a \subseteq^* -lower bound of $y \setminus \mathcal{G}$. Since \mathcal{G} generates $\mathcal{H} \cap y$, $z \in \mathcal{H}^{\perp}$; and since it is infinite, it witnesses that $y \notin \mathcal{H}^{\perp\perp}$. \square

The following is essentially the characterization of \mathfrak{b} as the least cardinal κ for which there exists an (ω, κ) gap in $\mathcal{P}(\omega) / \text{Fin}$.

Lemma 5.5 ($\mathfrak{b} > \lambda$). *Let $\mathcal{H} \subseteq [\theta]^{\leq \aleph_0}$. If \mathcal{H} is λ -generated then $(\mathcal{H}^\perp, \subseteq^*)$ is σ -directed.*

The following should be compared to the principle (A).

- (\star) Every directed subfamily \mathcal{H} of $([\theta]^{\leq \aleph_0}, \subseteq^*)$ for which $\downarrow \mathcal{H}$ is generated by a subset $\mathcal{B} \subseteq \mathcal{H}$ of cardinality $|\mathcal{B}| < \mathfrak{b}$, and is locally generated by fewer than \mathfrak{p} elements, has either
- (1) a countable decomposition of θ into singletons and sets locally in $\downarrow \mathcal{H}$,
 - (2) an uncountable subset of θ orthogonal to \mathcal{H} .

Proposition 5.6 ($\mathfrak{p} > \aleph_1$). (\star) *implies* (A).

Proof. Note that $\mathfrak{p} \leq \mathfrak{b}$. □

Theorem 5.1. (*) *implies* (\star).

Proof. Let \mathcal{H} be as in the hypothesis of (\star). By lemma 5.4, equation (58) is satisfied. And by lemma 5.5, \mathcal{H}^\perp is σ - \subseteq^* -directed. Therefore, (*) can be applied to \mathcal{H}^\perp .

First suppose that the second alternative (2) of (*) holds. Then there is a countable decomposition of θ into pieces orthogonal to \mathcal{H}^\perp , and therefore the first alternative of (*) holds by lemmas 2.6 and 5.4 since $\downarrow(\mathcal{H}, \subseteq^*) = \mathcal{H}^{\perp\perp}$.

Otherwise, (*) implies that the first alternative (1) of (*) holds, and thus there is an uncountable $X \subseteq \theta$ locally in $\downarrow(\mathcal{H}^\perp) = \mathcal{H}^\perp$. This means that X is orthogonal to \mathcal{H} . □

Corollary 5.1.1. $\mathfrak{p} > \aleph_1$ and (*) together imply (A).

Corollary 5.1.2 ($\mathfrak{p} > \aleph_1$). $(\star_{\max}) \rightarrow (\star) \rightarrow (\star_{\text{boc}}) \rightarrow (*) \rightarrow (\text{A})$.

In the article [Hir07c] we will establish that one of our combinatorial principles, in conjunction with $\mathfrak{p} > \aleph_1$, implies (A*). Note that it is already known (cf. [AT97]) that MA_{\aleph_1} and (A) imply (A*).

Of course, corollary 5.1.1 does not replace the principle (A). On the other hand, if it is the case that (A) implies $\mathfrak{p} > \aleph_1$, then this would render (A) obsolete. Hence we are interested in the following question (recall from section 1 that (A) $\rightarrow 2^{\aleph_0} > \aleph_1$).

Question 7. Does (A) imply $\mathfrak{p} > \aleph_1$? How about $\mathfrak{b} > \aleph_1$?

5.2. Nearly special Aronszajn trees. In [AH07] notions of ‘almost specialness’ for Aronszajn trees were considered. The main goal was to show that all Aronszajn trees being ‘almost special’ does not necessarily imply that all Aronszajn trees are special in the usual sense. The following notion was proposed and appears to be optimal with respect to being nearly special.

Definition 5.7. We say that an ω_1 -tree T is $(\text{NS}^+, \text{NS}^*)$ -special if there exists a maximal antichain \mathcal{A} of (NS^+, \subseteq) such that every $S \in \mathcal{A}$ has a relatively closed $C \subseteq S$ on which $T \upharpoonright C$ is special (cf. §2.1).

The following is the main result of [AH07].

Theorem (Abraham–Hirschorn). *The existence of a nonspecial Aronszajn tree is simultaneously consistent with CH and every Aronszajn tree being $(\text{NS}^+, \text{NS}^*)$ -special.*

We now show that the results of this paper do indeed generalize this theorem. Given a tree T , we assume without loss of generality that it resides on some ordinal θ . The associated family, defined in [AT97], is

$$\mathcal{I}_T = \{x \in [\theta]^{\leq \aleph_0} : x \perp \{\text{pred}_T(\xi) : \xi \in \theta\}\}, \quad (59)$$

in other words, $x \in [\theta]^{\leq \aleph_0}$ is in \mathcal{I}_T iff every node of the tree has only finitely many predecessors in x . Clearly \mathcal{I}_T is \subseteq^* -directed and is moreover a lower subset of $[\theta]^{\leq \aleph_0}$; indeed it is an ideal. As is well known (e.g. [AT97]), in the case $\theta = \omega_1$, if T is an ω_1 -tree then \mathcal{I}_T is $\sigma\text{-}\subseteq^*$ -directed and thus a P -ideal. More generally, the following lemma 5.8 follows. This embedability property was shown to be equivalent to what we called *local countability* in [Hir07b].

Lemma 5.8. *If T is a tree that embeds into an ω_1 -tree then \mathcal{I}_T is σ -directed.*

The following is observed in [AT97]. Lemma 5.9 implies that when X is locally in \mathcal{I}_T it is a countable union of antichains (i.e. special in our terminology).

Lemma 5.9. *If $X \subseteq \theta$ is locally in \mathcal{I}_T then as a subtree of T , X is of height at most ω .*

It is noted in [Tod00] that if X is orthogonal to \mathcal{I}_T , then X is a finite union of chains. We provide a proof here, using the following basic tree lemma, which to the best of our knowledge is from the folklore.

Lemma 5.10. *Let T be a tree. If T has no infinite antichains then it is a finite union of chains.*

Proof. Let \mathcal{C} be the collection of all maximal chains in T . We may assume that all levels of T are finite. Supposing that \mathcal{C} is infinite, T must have infinitely many splitting nodes. We choose $s_n, t_n \in T$ by recursion on $n < \omega$ so that each s_n is a splitting node and $s_n <_T t_{n+1}, s_{n+1}$ with $s_{n+1} \neq t_{n+1}$ on the same level. This is possible, because s_n is finitely splitting and thus we can recursively guarantee that $\{C \in \mathcal{C} : s_n \in C\}$ is infinite, which implies that there is another splitting node strictly above s_n . Then $\{t_1, t_2, \dots\}$ forms an infinite antichain of T . \square

Corollary 5.10.1. *If $X \subseteq \theta$ is orthogonal to \mathcal{I}_T then it is a finite union of chains.*

Proof. Note that \mathcal{I}_T includes the countable antichains of T . Thus X contains no infinite antichains, and lemma 5.10 is applied to the subtree (X, \leq_T) . \square

Theorem 5.2. $(*_s)$ *implies that every Aronszajn tree is $(\text{NS}^+, \text{NS}^*)$ -special.*

Proof. Let \mathcal{A} be the maximal antichain of NS^+ given by $(*_s)$. Let T be an Aronszajn tree, and enumerate each level T_α ($\alpha < \omega_1$) as $\xi_{\alpha,0}, \xi_{\alpha,1}, \dots$. For each n , define a tree (U, \leq_U) on ω_1 by

$$\alpha \leq_{U_n} \beta \quad \text{if} \quad \xi_{\alpha,n} \leq_T \xi_{\beta,n}. \quad (60)$$

Thus each U_n embeds into T via $\alpha \mapsto \xi_{\alpha,n}$. And thus each \mathcal{I}_{U_n} is σ -directed by lemma 5.8. By corollary 5.10.1, since T is Aronszajn, no \mathcal{I}_{U_n} can have an uncountable set orthogonal to it (let alone stationary). Therefore the second alternative (2) of $(*_s)$ gives for each $S \in \mathcal{A}$, and each n , a relatively closed $C_{S,n} \subseteq S$ locally in \mathcal{I}_{U_n} . Then each $C_{S,n}$ is special as a subtree of U_n by lemma 5.9, which means that $\{\xi_{\alpha,n} : \alpha \in C_{S,n}\}$ is special as a subtree of T . For each $S \in \mathcal{A}$, put $C_S = \bigcap_{n=0}^{\infty} C_{S,n}$. It is now clear that $T \restriction C_S$ is special, completing the proof. \square

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